Solutions to Prelim 1 5b) Intended and As Written

Problem 5) (25 Points)

Let
\[ x = u^2 - v^2 \]
\[ y = 2uv. \]

Suppose \( z = f(x, y) \) is a differentiable function.

5a) Express \( \frac{\partial z}{\partial u} \) in terms of \( \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, u, \) and \( v. \)

Solution:
\[
\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = 2u \frac{\partial z}{\partial x} + 2v \frac{\partial z}{\partial y}.
\]

Similarly, not needed til 5b):
\[
\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = (-2v) \frac{\partial z}{\partial x} + 2u \frac{\partial z}{\partial y}.
\]

5b Intended) Show that

This is a moderate difficulty problem similar to homework problem 38 from section 14.6

\[
\frac{1}{4(u^2 + v^2)} \left[ \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 \right] = \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2.
\]

5b As Written) Show that

Because of the typo, even though it is doable, this is too hard a problem for people to do without practice. Consequently it was not counted in grading the prelim. Sorry for the mistake!

\[
\frac{1}{4(u^2 + v^2)} \left( \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right) = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}.
\]

Solution to 5b) Intended: Using the formulas above in the 5a solution (including the \( z \) one)

\[
\frac{1}{4(u^2 + v^2)} \left( \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right) = \frac{1}{4(u^2 + v^2)} \left[ \left( 2u \frac{\partial z}{\partial x} + 2v \frac{\partial z}{\partial y} \right)^2 + \left( -2v \frac{\partial z}{\partial x} + 2u \frac{\partial z}{\partial y} \right)^2 \right]
\]

Since the cross term \( \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \) shows up twice on the right hand side with opposite signs, these terms cancel, and the right hand simplifies to

\[
\frac{1}{4(u^2 + v^2)} \left[ (4u^2 + 4v^2) \left( \frac{\partial z}{\partial x} \right)^2 + (4u^2 + 4v^2) \left( \frac{\partial z}{\partial y} \right)^2 \right]
\]
which further simplifies to the desired

\[ \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2. \]

**Solution to 5b) As Written:** The most important additional idea needed here is to realize that the task (in terms of the chain rule) of differentiating (with respect to e.g. \( u \)) an expression like \( \frac{\partial z}{\partial x} \) is just like the task of differentiating \( z(x, y) \). So using the exact same argument as in the solution to 5a, we quickly have

\[
\begin{align*}
\frac{\partial}{\partial u} \left( \frac{\partial z}{\partial x} \right) &= 2u \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) + 2v \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = 2u \frac{\partial^2 z}{\partial x^2} + 2v \frac{\partial^2 z}{\partial x \partial y} \\
\frac{\partial}{\partial v} \left( \frac{\partial z}{\partial x} \right) &= (-2v) \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) + 2u \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = -2v \frac{\partial^2 z}{\partial x^2} + 2u \frac{\partial^2 z}{\partial x \partial y} \\
\frac{\partial}{\partial u} \left( \frac{\partial z}{\partial y} \right) &= 2u \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) + 2v \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = 2u \frac{\partial^2 z}{\partial x \partial y} + 2v \frac{\partial^2 z}{\partial y^2} \\
\frac{\partial}{\partial v} \left( \frac{\partial z}{\partial y} \right) &= (-2v) \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) + 2u \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = -2v \frac{\partial^2 z}{\partial x \partial y} + 2u \frac{\partial^2 z}{\partial y^2}.
\end{align*}
\]

Then using the solution to 5a) again,

\[
\begin{align*}
\frac{\partial}{\partial u} \left( \frac{\partial z}{\partial u} \right) &= 2 \frac{\partial z}{\partial x} + 2 \frac{\partial z}{\partial y} \\
\frac{\partial}{\partial u} \left( 2u \frac{\partial z}{\partial x} + 2v \frac{\partial z}{\partial y} \right) &= 2 \frac{\partial z}{\partial x} + 2u \left( \frac{\partial}{\partial u} \left( \frac{\partial z}{\partial x} \right) \right) + 2v \left( \frac{\partial}{\partial u} \left( \frac{\partial z}{\partial y} \right) \right) \\
&= 2 \frac{\partial z}{\partial x} + 2u \left( \frac{\partial}{\partial u} \left( \frac{\partial z}{\partial x} \right) \right) + 2v \left( \frac{\partial}{\partial u} \left( \frac{\partial z}{\partial y} \right) \right) \\
&= 2 \frac{\partial z}{\partial x} + 2u \left( \frac{\partial^2 z}{\partial x^2} + 2v \frac{\partial^2 z}{\partial x \partial y} \right) + 2v \left( \frac{\partial^2 z}{\partial x \partial y} + 2u \frac{\partial^2 z}{\partial y^2} \right)
\end{align*}
\]

which simplifies to

\[ \frac{\partial^2 z}{\partial u^2} = 2 \frac{\partial z}{\partial x} + 4u^2 \frac{\partial^2 z}{\partial x^2} + 8uv \frac{\partial^2 z}{\partial x \partial y} + 4v^2 \frac{\partial^2 z}{\partial y^2}. \]

Similarly one finds (in part because \( \frac{\partial}{\partial v} \left( (-2v) \frac{\partial z}{\partial x} + 2u \frac{\partial z}{\partial y} \right) \) has a \(-2 \frac{\partial z}{\partial x}\) term) that

\[ \frac{\partial^2 z}{\partial v^2} = -2 \frac{\partial z}{\partial x} + 4v^2 \frac{\partial^2 z}{\partial x^2} - 8uv \frac{\partial^2 z}{\partial x \partial y} + 4u^2 \frac{\partial^2 z}{\partial y^2}. \]

Adding the two expressions quickly gives the desired result.