Math 2130 Spring 2013 Final Exam

1. Is there a function f that has the following partial derivatives?

$$\frac{\partial f}{\partial x} = 4x^3y^2 - 3y^4,$$
$$\frac{\partial f}{\partial y} = 2x^4y - 12xy^3.$$

If so, find f. If not, quote the test or theorem that implies no such f can exist.

2. Find an equation of the tangent plane to the surface $x^2 = y^3 - z$ at the point $(1,1,0)$.

3. Find the volume of the region bounded by the planes $z = 3y$, $z = y$, $y = 1$, $x = 1$, and $x = 2$.

4. Imagine you are riding a bicycle along a straight flat road late at night. To help cars see you, you have attached a safety light to your right ankle. The goal of this question is to find parametric equations for the path traced out by the safety light. Assume that your bike moves at a speed of 25 km/hr and your ankle moves in a circle of radius 20 cm centered 30 cm above the ground, making one revolution per second. Find parametric equations for $x(t)$ and $y(t)$, which describe the path traced out by the light.

5. Consider the line integral $\oint_C \left(y\,\vec{i} + x^2\vec{j}\right)\cdot d\vec{r}$ where C is the counterclockwise path around the perimeter of the rectangle $0 \le x \le 2$, $0 \le y \le 3$.
a) Use Green's theorem to evaluate the line integral.
b) Check your answer to (a) by evaluating the line integral directly.

6. You move to a town where two of your most annoying childhood "friends" live. To avoid seeing them, you decide to rent an apartment in an optimal location (x, y), your criterion being that the sum of the squares of the distances between your apartment and each of theirs should be made as large as possible. Given that one of them lives at (a_1, b_1) and the other at (a_2, b_2), and that the town is circular so your location must satisfy the constraint $x^2 + y^2 \le 1$, where is the optimal place for you to live? (You may be able to solve this by simple geometry and common sense, but for full credit, please use the methods you learned in this course.)

7. Let $\vec{F}(x,y,z) = y\vec{i} + x\vec{j} + y\vec{k}$.

a) Show that the vector field $\vec{F}$ is path dependent.

b) Nevertheless, there are certain closed paths C in 3-dimensional space for which $\oint_C \vec{F} \cdot d\vec{r} = 0$. Find one such path. (Please justify your answer on theoretical grounds or by performing any necessary calculations.)

8. This is a question about the geometric and Cartesian coordinate definitions of divergence. Let $\vec{F}(x,y,z) = (x + x^2)\vec{i}$.

a) Calculate div $\vec{F}$ at the origin using partial derivatives (the Cartesian coordinate definition).

The goal of the rest of the problem is to show that your answer to part (a) agrees with the geometric definition of the divergence, as follows.

b) Find the flux of $\vec{F}$ out of a tiny cube with sides of length c, one corner at the origin, and three of its sides lying along the x, y, and z axes.

c) Use your answer to part (b) and the geometric definition of divergence to calculate the divergence of $\vec{F}$ at the origin.