Math 2130: The 2-Dimensional Divergence Theorem

Water is incompressible: for any region entirely filled with water the amount of water flowing in is equal to the amount of water flowing out. This has many practical applications: we can design plumbing systems, solve for the shape water forms when flowing out of a faucet, or approximately relate the flow through a waterfall with the flow of water downstream.

Water typically flows through three dimensional regions, but we will start with the divergence theorem for two dimensions, since two dimensional regions are easier to visualize. We will begin by working out a way of measuring the amount of a vector field flowing through a line in two dimensions.

**Definition.** The flux integral of a 2 dimensional vector field \( \vec{F}(x,y) = (F_1(x,y), F_2(x,y)) \) across a curve \( C \) parameterized by \( \vec{r}(t) \) for \( a \leq t \leq b \) is given by:

\[
\int_C (-F_2, F_1) \cdot d\vec{r} = \int_{t=a}^{t=b} (-F_2(\vec{r}(t)), F_1(\vec{r}(t))) \cdot \vec{r}'(t) \, dt.
\]

Here we take advantage of the fact that we already know how to compute how much of a vector field is flowing along the direction of a curve using vector line integrals\(^1\). If we rotate the vectors of \( \vec{F} \) by 90 degrees counterclockwise:

- Vectors pointing to the right of the curve become vectors pointing along the curve and are counted positively in our flux integral.
- Vectors pointing along the curve become vectors pointing to the left of the curve and are not counted in our flux integral.
- Vectors pointing to the left of the curve become vectors pointing backwards along the curve and are counted negatively in our flux integral.
- Vectors pointing backwards along the curve become vectors pointing to the right of the curve and are not counted in our flux integral.
- If vectors have components to the left/right and along/backwards along the curve, the components to the left and right get counted in our flux integral, and components along/backwards along the curve don’t get counted in our flux integral.

We now need a way of describing “incompressible” vector fields. We will use divergence, a topic covered in section 19.3 of the textbook. You should read that if you haven’t yet before moving on.

Recall that the divergence of a vector field \( \vec{F}(x,y) = (F_1(x,y), F_2(x,y)) \) is given by:

\[ \nabla \cdot \vec{F} = (F_1)_x + (F_2)_y \]

---

\(^1\)The most common notation used for flux integrals is \( \oint_C \vec{F} \cdot \vec{n} \, ds \). In this case, instead of rotating \( \vec{F} \) counterclockwise by 90 degrees, we rotate \( d\vec{r} \) by 90 degrees clockwise. Note that \( \frac{d\vec{r}}{dt} = \frac{\vec{r}'(t)}{||\vec{r}'(t)||} ||\vec{r}'(t)|| dt \), so \( \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \frac{\vec{r}'(t)}{||\vec{r}'(t)||} ||\vec{r}'(t)|| dt \). The vector \( \frac{\vec{r}'(t)}{||\vec{r}'(t)||} \) is called \( \vec{T} \) and is a unit vector in the direction of \( C \). The vector \( \vec{n} \) is \( \vec{T} \) rotated 90 degrees clockwise. We won’t use this notation in this class.
If we view a vector field as describing the velocity of particles in a liquid or gas, the divergence is measuring how much the particles are moving away from each other, with negative divergence meaning the particles are moving towards each other. In the case of an incompressible velocity field, the divergence is 0 everywhere.

We now can write our assertion that the water flowing out of a region minus the water flowing into a region is 0:

\[ \int_C (-F_2, F_1) \cdot d\vec{r} = 0 \]

Note that since we parameterized the loop counterclockwise, vectors pointing out of the region count positively and vectors pointing into the region count negatively. Our assertion above is that these cancel.

If \( \nabla \cdot \vec{F} > 0 \) somewhere, we can think of it as a region where water is being added, and if \( \nabla \cdot \vec{F} < 0 \) somewhere, we can think of it as a region where water is being drained away. As such, if we add up all of the positive divergence and subtract when we have negative divergence, we get the net amount of water flowing into the interior of the region.

The full Divergence Theorem says that the total amount of water flowing out of a region minus the total amount flowing into the region is the sum of the total amount of positive divergence minus the total amount of negative divergence:

\[ \int_C (-F_2, F_1) \cdot d\vec{r} = \int_R (\nabla \cdot \vec{F}) \, dA \]

**Example.** Let \( R \) be the solid unit circle and \( C \) the counterclockwise unit circle. Let \( \vec{F}(x, y) = (x, y) \). Verify the divergence theorem manually.

We first evaluate the left hand side:

\[
\begin{align*}
\int_C (-F_2, F_1) \cdot d\vec{r} &= \int_{t=0}^{t=2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t) \, dt \\
&= \int_{t=0}^{t=2\pi} 1 \, dt \\
&= 2\pi
\end{align*}
\]
Next we evaluate the right hand side:

$$\int_{R} (\nabla \cdot \vec{F}) dA = \int_{R} \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) dA$$

$$= \int_{R} 2 dA$$

$$= 2 \text{ Area}(R)$$

$$= 2 \pi$$

**Example.** Let $R$ be the square with vertices $(0,0), (2,0), (2,2)$ and $(0,2)$, and $C$ be its boundary oriented counterclockwise. Let $\vec{F}(x,y) = (2 - y, xy)$. Verify the divergence theorem manually. A picture is shown below:

![Picture of vectors](image)

We can use our intuition about flux and divergence to guess what we will find. It looks like vectors in the square are pulling away from each other: vectors towards the top right of the square are moving outwards more than the vectors towards the bottom left are moving inwards. So the divergence is probably positive. The flux over the bottom edge should be zero: the vector field is in the same direction as the curve. The flux over the right edge should be positive: a component of the vector field is pointing outwards (to the right). The flux over the top edge should also be positive: the vector field is pointing entirely outwards. The flux over the left edge however should be negative: the vector field is pointing inwards. The outwards-pointing vectors look larger and more numerous than the inward pointing vectors, so we can guess that the overall (net) flux should be positive.

The actual divergence is $\nabla \cdot \vec{F} = 0 + x = x$ which is positive over the square. Integrating

$$\int_{R} x dA = \int_{x=0}^{x=2} \int_{y=0}^{y=2} x \, dy \, dx$$

gives us 4.

We parameterize the bottom edge by $\vec{r}(t) = (t,0)$ for $0 \leq t \leq 2$. Integrating:

$$\int_{C_{\text{bottom}}} (-F_2, F_1) \cdot d\vec{r} = \int_{t=0}^{t=2} (0, 2) \cdot (1, 0) \, dt = 0.$$
We parameterize the right edge by \( \vec{r}(t) = (2, t) \) for \( 0 \leq t \leq 2 \). Integrating:

\[
\int_{C_{\text{right}}} (-F_2, F_1) \cdot d\vec{r} = \int_{t=0}^{t=2} (-2t, 2-t) \cdot (0, 1) \, dt = 2.
\]

We parameterize the top edge by \( \vec{r}(t) = (2 - t, 2) \) for \( 0 \leq t \leq 2 \). Integrating:

\[
\int_{C_{\text{top}}} (-F_2, F_1) \cdot d\vec{r} = \int_{t=0}^{t=2} -(2 - t)2, 0) \cdot (-1, 0) \, dt = 4.
\]

We parameterize the left edge by \( \vec{r}(t) = (0, 2 - t) \) for \( 0 \leq t \leq 2 \). Integrating:

\[
\int_{C_{\text{left}}} (-F_2, F_1) \cdot d\vec{r} = \int_{t=0}^{t=2} (0, t) \cdot (0, -1) \, dt = -2.
\]

Interestingly, notice that the flux into the left edge and flux out of the right edge cancel exactly. While the vectors on the left edge are smaller, they’re exactly perpendicular to the edge. Meanwhile the vectors on the right edge are longer, but at a very small angle to the edge. The overall flux integral is 4, which agrees with what we got when we integrated the divergence over the square.