1. (10 pts) Change the order of integration and evaluate $\int_{0}^{4} \int_{\sqrt{y}}^{2} \sqrt{2+x^3} \, dx \, dy$. 
2. (12 pts) Use the method of Lagrange multipliers to find the maximum area of a rectangle inscribed (as shown in the image below) in the ellipse \( \frac{x^2}{4} + y^2 = 1 \).
3. Short Answer. The parts of this question are unrelated.

(a) (4pts) Below are the level sets for a function $f(x,y,z)$, for values $c = -1, 0$ and $1$. Put an $x$ in the box corresponding to the correct function from those listed, and then circle the appropriate $c$ value for each level set.

\[
\begin{align*}
\text{f}(x,y,z) &= x^2 + y^2 - z^2 & \text{f}(x,y,z) &= x^2 + y^2 + z^2 \\
\text{f}(x,y,z) &= x^2 - y^2 + z^2 & \text{f}(x,y,z) &= -x^2 + y^2 + z^2
\end{align*}
\]

\[
\begin{array}{ccc}
\text{c} = -1 & 0 & 1 \\
\end{array}
\]

(b) (4 pts) Circle the best answer for each of the following.

\[
\begin{array}{|c|c|c|}
\hline
\text{i. Curl(Curl(F))} & \text{always zero} & \text{not always zero} & \text{not defined} \\
\hline
\text{ii. Div(\nabla f)} & \text{always zero} & \text{not always zero} & \text{not defined} \\
\hline
\text{iii. Curl(\nabla f)} & \text{always zero} & \text{not always zero} & \text{not defined} \\
\hline
\text{iv. Curl(Div(F))} & \text{always zero} & \text{not always zero} & \text{not defined} \\
\hline
\end{array}
\]
(c) (4pts) Consider the surface S parameterized by \( G(u,v) = (\sqrt{1+u^2}\cos v, \sqrt{1+u^2}\sin v, u) \) for \(-1 \leq u \leq 1\) and \(0 \leq v \leq 2\pi\). Make an ‘X’ in the box corresponding to the picture of S.

![Picture Options]

(d) (6pts) Let D be a region in the plane enclosed by a simple closed curve. Using Green’s Theorem, we can compute the area of D by integrating a vector field F around the boundary of D. Which, if any, of the following vector fields could you use for this purpose?

- \( F = \langle x^2 + y^2, 2x + 3y \rangle \)
- \( F = \langle x, y \rangle \)
- \( F = \langle 2x\sin y, x + x^2\cos y \rangle \)
- None of these give the area

Check the appropriate boxes, and give a justification for your answer below.
(e) (6 pts) The vector field $\mathbf{F}$ shown below is conservative. To give an idea of the scale, $\|\mathbf{F}(0,y)\| = \frac{1}{2}$

Circle the correct answer.

$\int_C \mathbf{F} \cdot d\mathbf{r} =$

$\begin{array}{ccccccc}
-4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\
\end{array}$

Carefully justify your answer below.
4. (8 pts) Below is a map of Drydan lake. Let D be the region in the xy plane corresponding to the lake. Let \( f(x,y) \) be the depth in feet of the lake at a point \((x,y)\). Some level curves for \( f \) are shown. The boundary of D is the outermost unlabeled curve.

(a) Assume that each box has length and width equal to 500 ft. Using the sample points shown in the grid, find an approximation for \( \iint_D f(x,y) \, dA \). Show how you arrived at your answer. You do not need to simplify.

(b) In the context of this question, what does \( \iint_D f(x,y) \, dA \) represent? What would the units be?
5. (12 pts) Two surfaces are said to be orthogonal to each other at a point \( P \) if the normals to their tangent planes are orthogonal. Show that the surfaces

\[
z = \frac{1}{2}(x^2 + y^2 - 1) \quad \text{and} \quad z = \frac{1}{2}(1 - x^2 - y^2)
\]

are orthogonal at all points of intersection. Be sure that your answer contains a clear description of the points of intersection. Note that the following page in essentially blank, in case you need more space for your answer to this question.
More space for your answer to question 5.
6. (14 pts) Consider the function \( f(x, y) = y - y \sin x \). The graph of \( f(x, y) \) is a surface. Assume that there is a bug on the surface at the point \( (\frac{3\pi}{2}, 2, 4) \). This question contains 4 parts, two on this page, and two on the following page.

(a) Which direction, in the xy-plane, should the bug walk to go uphill the fastest?

(b) Now suppose the bug wants to walk along the contour \( f(x, y) = 4 \). Which direction (or directions) should it initially head?
(c) Now assume the bug is walking northwest. What is the initial rate of change of the altitude of the bug, with respect to the horizontal distance? Is the bug walking up or downhill?

(d) Again, assume that the bug is walking northwest, and its horizontal (in the xy-direction) speed is 2 centimeters per second. What is the initial rate of change of the altitude with respect to time? Give a justification for your answer.
7. (24 pts) Let $S$ be the surface shown in the figure to the right, oriented by the outward pointing normal shown. The boundary of $S$ is the circle with radius 3 centered about the origin in the $xz$-plane. Recall that the course convention is to label the positive axes with $x$, $y$, and $z$.

(a) The orientation of $S$ induces an orientation on $\partial S$. Draw an arrow on $\partial S$ showing this orientation.

(b) Let $\mathbf{F} = (z + x^2 y, x y^2 z, -x + 1)$. Calculate the $\text{Curl}(\mathbf{F})$.

\[
\text{Curl}(\mathbf{F}) =
\]

(c) Let $D$ be the disk centered at the origin in the $xz$-plane that shares a boundary with $S$. Together, $S$ and $D$ bound a solid $W$. Let $\mathbf{n}$ be the outward (from $W$) pointing unit normal to $D$. What is $\mathbf{n}$?

\[
\mathbf{n} = 
\]
(d) Recall \( \mathbf{F} = (z + x^2y, xy^2z, -x + 1) \). Use Stokes' Theorem to compute:

\[
\iint_S \text{Curl}(\mathbf{F}) \cdot \mathbf{dS}.
\]
(e) Now let $\mathbf{G} = (2, 3, -4)$. Calculate the flux of $\mathbf{G}$ through $S$. Carefully justify your answer.