MATH 192 SUMMER 2007 PRELIM 2

SOLUTIONS

1. (8 pts) Consider the the function $f(x,y) = x^2 + xy + y^2$.

   (a) Find the directions in which $f(x,y)$ increases and decreases most rapidly at $P_0(-1,1)$.
   Solution. $f(x,y)$ increases most rapidly at $P_0(-1,1)$ in the direction of $\nabla f(-1,1)$ and decreases most rapidly at $P_0(-1,1)$ in the direction of $-\nabla f(-1,1)$. We have

$$\frac{\partial f}{\partial x}(-1,1) = (2x+y)\Big|_{(x,y)=(-1,1)} = 2(-1)+1 = -1,$$
$$\frac{\partial f}{\partial y}(-1,1) = (x+2y)\Big|_{(x,y)=(-1,1)} = -1+2 = 1.$$

   Hence $\nabla f(-1,1) = -\mathbf{i}+\mathbf{j}$ and $\nabla f(-1,1) = \mathbf{i}-\mathbf{j}$.

   (b) Find the derivatives of $f(x,y)$ in these directions.
   Solution. Consider the unit vector

$$\mathbf{u} = \frac{\nabla f(-1,1)}{|\nabla f(-1,1)|} = \frac{-\mathbf{i}+\mathbf{j}}{|-\mathbf{i}+\mathbf{j}|} = \frac{-\mathbf{i}+\mathbf{j}}{\sqrt{2}}$$

The derivative of f in the direction of $\nabla f(-1,1)$ is

$$D_{\mathbf{u}}f(-1,1) = \nabla f(-1,1)\cdot\mathbf{u} = (-\mathbf{i}+\mathbf{j})\cdot\frac{-\mathbf{i}+\mathbf{j}}{|-\mathbf{i}+\mathbf{j}|} = |-\mathbf{i}+\mathbf{j}| = \sqrt{2},$$

while the derivative of f in the direction of $-\nabla f(-1,1)$ is

$$D_{-\mathbf{u}}f(-1,1) = \nabla f(-1,1)\cdot(-\mathbf{u}) = (-\mathbf{i}+\mathbf{j})\cdot\big(-\frac{-\mathbf{i}+\mathbf{j}}{|-\mathbf{i}+\mathbf{j}|}\big) = -|-\mathbf{i}+\mathbf{j}| = -\sqrt{2}.$$

2. (7 pts) Find the linearization $L(x,y,z)$ of $f(x,y,z) = \dfrac{\sin(xy)}{z}$ at $(\frac{\pi}{2},1,1)$.
Solution. We have

$$f\left(\frac{\pi}{2},1,1\right) = \frac{\sin(\frac{\pi}{2})\cdot 1}{1} = 1$$
$$\frac{\partial f}{\partial x}\left(\frac{\pi}{2},1,1\right) = \frac{y\cos(xy)}{z}\Big|_{(x,y,z)=(\frac{\pi}{2},1,1)} = \frac{1\cdot\cos(\frac{\pi}{2})}{1} = 0$$
$$\frac{\partial f}{\partial y}\left(\frac{\pi}{2},1,1\right) = \frac{x\cos(xy)}{z}\Big|_{(x,y,z)=(\frac{\pi}{2},1,1)} = \frac{\frac{\pi}{2}\cos(\frac{\pi}{2})}{1} = 0$$
$$\frac{\partial f}{\partial z}\left(\frac{\pi}{2},1,1\right) = \frac{-\sin(xy)}{z^2}\Big|_{(x,y,z)=(\frac{\pi}{2},1,1)} = \frac{-\sin(\frac{\pi}{2})}{1^2} = -1$$

Therefore, the linearization of $f(x,y,z) = \dfrac{\sin(xy)}{z}$ at $(\frac{\pi}{2},1,1)$ is

$$\begin{aligned} L(x,y,z) &= f\left(\frac{\pi}{2},1,1\right) + \frac{\partial f\left(\frac{\pi}{2},1,1\right)}{\partial x}(x-\frac{\pi}{2}) + \frac{\partial f\left(\frac{\pi}{2},1,1\right)}{\partial y}(y-1) + \frac{\partial f\left(\frac{\pi}{2},1,1\right)}{\partial z}(z-1) \\ &= 1 + 0\cdot(x-\frac{\pi}{2}) + 0\cdot(y-1) + (-1)\cdot(z-1) \\ &= 1-(z-1) \\ &= \boxed{2-z} \end{aligned}$$

3. (7 pts) Find the critical point(s) of $f(x,y) = x^2+y^2+xy+1$ and determine their nature. Justify your answer.
Solution. We have

$$\frac{\partial f(x,y)}{\partial x} = 2x+y \quad \text{and} \quad \frac{\partial f(x,y)}{\partial y} = 2y+x,$$

for all $(x,y)\in\mathbb{R}^2$. Since the partial derivatives exist everywhere, in order to find the critical points of f we need only solve the system

$$2x+y=0 \quad \text{and} \quad 2y+x=0.$$

It is easy to see that the only solution to this system is the pair $(x,y) = (0,0)$ (check that!). The Hessian of f is equal to

$$\begin{vmatrix} \dfrac{\partial^2 f(x,y)}{\partial x^2} & \dfrac{\partial^2 f(x,y)}{\partial x\partial y} \\ \dfrac{\partial^2 f(x,y)}{\partial y\partial x} & \dfrac{\partial^2 f(x,y)}{\partial x^2} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 > 0,$$

for all $(x,y)\in\mathbb{R}^2$. Since

$$\frac{\partial^2 f(0,0)}{\partial x^2} = 2 > 0,$$

$f(0,0)=1$ is a local minimum value of f on $\mathbb{R}^2$. Since $\mathbb{R}^2$ is open and $(0,0)$ is the only critical point of f, $f(0,0)=1$ is the *absolute* minimum value of f on $\mathbb{R}^2$.

4. (8 pts) Find the extrema of $x^2y^2z^2$ on the sphere $x^2+y^2+z^2=3$.
Solution. Consider the function $f(x,y,z)=x^2y^2z^2$. In order to find the extrema of f on the given sphere, we find the values of x,y,z and λ that simultaneously satisfy the equations

$$\begin{aligned} \nabla f(x,y,z) &= \lambda\nabla g(x,y,z), \\ g(x,y,z) &= 0, \end{aligned}$$

where $g(x, y, z) = x^2 + y^2 + z^2 - 3$. Since

$$\nabla f(x, y, z) = 2xy^2z^2\mathbf{i} + 2x^2yz^2\mathbf{j} + 2x^2y^2z\mathbf{k}$$

and

$$\nabla g(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k},$$

we must solve the following system

$$xy^2z^2 = \lambda x, \quad (1)$$
$$x^2yz^2 = \lambda y, \quad (2)$$
$$x^2y^2z = \lambda z, \quad (3)$$
$$x^2 + y^2 + z^2 = 3. \quad (4)$$

We examine the following two cases

Case 1. $xyz = 0$.
If $x = 0$, then (2) and (3) imply that $\lambda y = \lambda z = 0$. This can only happen if $\lambda = 0$, for otherwise we would have $y = z = 0$ contradicting (4). Then y and z satisfy the equation $y^2 + z^2 = 3$. Similarly, if $y = 0$, then $\lambda = 0$ and $x^2 + z^2 = 3$, while if $z = 0$, then $\lambda = 0$ and $x^2 + y^2 = 3$.

Case 2. $xyz \neq 0$.
In this case $\lambda \neq 0$, and so we may divide (1) by (2) and (2) by (3) to obtain

$$\frac{y}{x} = \frac{x}{y} \quad \text{and} \quad \frac{z}{y} = \frac{y}{z},$$

respectively. From these two equations along with (4), it follows that

$$x^2 = y^2 = z^2 = 1 \quad \text{and} \quad \lambda = 1.$$

The value of f at any point of the sets

$$\{(x, y, z) \in \mathbb{R}^3 \mid x = 0,\ y^2 + z^2 = 3\}$$
$$\{(x, y, z) \in \mathbb{R}^3 \mid y = 0,\ x^2 + z^2 = 3\}$$
$$\{(x, y, z) \in \mathbb{R}^3 \mid z = 0,\ x^2 + y^2 = 3\}$$

is 0, while the value of f at any point of the set

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 = y^2 = z^2 = 1\}.$$

is 1. Since f is continuous on the given sphere, which is a closed and bounded set, f has a minimum and a maximum value on the sphere. Consequently, the extrema of f are 0 and 1.

Remark. 1.The solution set to the above system is the union of the following sets

$$S_1 = \{(x, y, z, \lambda)|\, x = 0,\ y^2 + z^2 = 3, \lambda = 0\}$$
$$S_2 = \{(x, y, z, \lambda)|\, y = 0,\ x^2 + z^2 = 3, \lambda = 0\}$$
$$S_3 = \{(x, y, z, \lambda)|\, z = 0,\ x^2 + y^2 = 3, \lambda = 0\}$$
$$S_4 = \{(x, y, z, \lambda)|\, x^2 = y^2 = z^2 = 1, \lambda = 1\}$$

2. For the minimum value of f we could just argue as follows: Since $f(x, y, z) \geq 0$ for all $(x, y, z) \in \mathbb{R}^3$ and $f(\sqrt{3}, 0, 0) = 0$, the minimum value of f on the sphere $x^2 + y^2 + z^2 = 3$ is 0. In this way, we could only examine the second case where $xyz \neq 0$.

5. (7 pts) Evaluate the double integral $\displaystyle\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin(y)}{x+y}\, dy\, dx$.
Hint: $\ln(a) - \ln(b) = \ln(a/b)$ $(a, b > 0)$.
Solution. In order to compute the integral in question, with the given order of integration, we must find

$$\int \frac{\sin(y)}{x+y}\, dy.$$

This cannot be computed by an elementary method, so we change the order of integration. To do this, we first rewrite the region of integration

$$R = \{(x, y)|\, 0 \leq x \leq \frac{\pi}{2}, \quad x \leq y \leq \frac{\pi}{2}\}$$
$$= \{(x, y)|\, 0 \leq y \leq \frac{\pi}{2}, \quad 0 \leq x \leq y\}$$

By changing the order of integration, the integral in question becomes

$$\begin{aligned}
\int_0^{\pi/2}\int_0^{y} \frac{\sin(y)}{x+y}\, dx\, dy &= \int_0^{\pi/2} \sin(y)\Big[\ln(x+y)\Big]_{x=0}^{x=y} dy \\
&= \int_0^{\pi/2} \sin(y)(\ln(2y) - \ln(y))\, dy \\
&= \ln(2)\int_0^{\pi/2} \sin(y)\, dy \\
&= \ln(2)\Big[-\cos(y)\Big]_{y=0}^{y=\pi/2} \\
&= \ln(2)\,(0 - (-1)) \\
&= \boxed{\ln(2)}.
\end{aligned}$$

6. (7 pts) Evaluate the double integral

$$\iint_R \sqrt{1-x^2-y^2}\,dA,$$

where R is the unit disk centered at the origin.

Solution. The polar form of the unit disk centered at the origin is

$$R = \{(r,\theta)|\, 0 \le r \le 1, \quad 0 \le \theta \le 2\pi\}.$$

Since $x^2 + y^2 = r^2$, the integral in question is equal to

$$\begin{aligned}\int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\,r dr\,d\theta &= \int_0^{2\pi}\int_1^0 \frac{-\sqrt{u}}{2} du\,d\theta \quad (u = 1-r^2, du = -2r\,dr)\\ &= \int_0^{2\pi} \left[\frac{-u^{3/2}}{3}\right]_{u=1}^{u=0} d\theta\\ &= \int_0^{2\pi} \frac{1}{3}\,d\theta\\ &= \frac{1}{3}(2\pi - 0)\\ &= \boxed{\frac{2\pi}{3}}.\end{aligned}$$

7. (7 pts) Express the volume of the pyramid D with vertices $O(0,0,0)$, $A(1,0,0)$, $B(1,1,0)$ and $C(0,0,1)$ as a ***triple*** integral. Set up the limits of integration but **do not evaluate** the integral.

Solution. The "lower" bounding surface of D lies in the xy-plane. Indeed, the "shadow" R of the pyramid D on the xy-plane is the triangle AOB. The line OB has equation $y = x$, and so we have

$$\begin{aligned}R &= \{(x,y)|\, 0 \le x \le 1, \quad 0 \le y \le x\}\\ &= \{(x,y)|\, 0 \le y \le 1, \quad y \le x \le 1\}\end{aligned}$$

The "upper" bounding surface of D lies in the plane $x + z = 1$ (check that!). Therefore,

$$\begin{aligned}V &= \int_0^1\int_0^x\int_0^{1-x} dz\,dy\,dx\\ &= \int_0^1\int_y^1\int_0^{1-x} dz\,dx\,dy\end{aligned}$$

Remarks. 1. The first integral expression for V seems easier to compute.

2. We obtain more formulas for V by considering the "shadow" of the pyramid D on the xz-plane, that is, the triangle AOC.

8. (7 pts) Evaluate the triple integral

$$\iiint_D \sqrt{x^2+y^2+z^2}\, dV,$$

where D is the unit sphere centered at the origin.
Solution. In spherical coordinates, the unit sphere centered at the origin is

$$D = \{(\rho, \varphi, \theta)\,|\, 0 \le \rho \le 1,\ 0 \le \varphi \le \pi,\ 0 \le \theta \le 2\pi\}.$$

Since $x^2 + y^2 + z^2 = \rho^2$, the integral in question is equal to

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho\,\rho^2 \sin(\varphi) d\rho\, d\varphi\, d\theta &= \int_0^{2\pi}\int_0^{\pi} \frac{\sin(\varphi)}{4}\, d\varphi\, d\theta \\
&= \int_0^{2\pi} \left[\frac{-\cos(\varphi)}{4}\right]_{\varphi=0}^{\varphi=\pi} d\theta \\
&= \int_0^{2\pi} \frac{2}{4}\, d\theta \\
&= \frac{1}{2}(2\pi - 0) \\
&= \boxed{\pi}.
\end{aligned}$$

9. (7 pts) Evaluate $\int_C (x^2+y^2+z^2)\, ds$, where C is the straight line segment from $(1,0,0)$ to $(1,3,0)$.
Solution. Consider the continuous function $f(x,y,z) = x^2+y^2+z^2$ over C. The given straight line segment passes through the point $(1,0,0)$ and is parallel to the vector $3\mathbf{j}$. The parametric equations of C are

$$x = 1, \quad y = 3t, \quad z = 0, \quad 0 \le t \le 1.$$

Thus, a parametrization of C is

$$\mathbf{r}(t) = \mathbf{i} + 3t\,\mathbf{j}, \quad 0 \le t \le 1,$$

which is smooth since $|\mathbf{r}'(t)| = 3 \ne 0$ for all $0 \le t \le 1$. Therefore

$$\begin{aligned}
\int_C (x^2+y^2+z^2)\, ds &= \int_C f(1, 3t, 0)|\mathbf{r}'(t)|\, dt \\
&= \int_0^1 3(1+9t^2)\, dt \\
&= 3\Big[t + 3t^3\Big]_{t=0}^{t=1} \\
&= 3(4-0) \\
&= \boxed{12}.
\end{aligned}$$