Math 192 Prelim 2 Solutions Spring 2006

1. i) $f(x,y,z) = \cos xy + e^{yz} + \ln xz \rightarrow \nabla f = (-y\sin xy + \frac{1}{x})\vec{i} + (-x\sin xy + ze^{yz})\vec{j} + (ye^{yz} + \frac{1}{z})\vec{k} \rightarrow \nabla f(1,0,1/2) = \vec{i}+\frac{1}{2}\vec{j}+2\vec{k}$, $\vec{u} = \vec{PQ}/|\vec{PQ}| = (\vec{i}+2\vec{j}+2\vec{k})/3$. Therefore, $D_{\vec{u}}f|_P = \nabla f \cdot \vec{u} = 1/3+1/3+4/3 = 2$.

ii) f increases most rapidly in the direction of ∇f and the value of the derivative in that direction is $|\nabla f| = \sqrt{1+1/4+4} = \sqrt{21}/2$.

2. Solve $(a\sqrt{2})^2 = 4a^2\cos 2\theta \rightarrow 1/2 = \cos 2\theta \rightarrow \theta = \pi/6$. Area $= 4\int_0^{\pi/6}\int_{a\sqrt{2}}^{2a\sqrt{\cos 2\theta}} r\,dr d\theta =$ $2\int_0^{\pi/6}(4a^2\cos 2\theta - 2a^2)\,d\theta = 2a^2[2\sin 2\theta - 2\theta]_0^{\pi/6} = 2a^2(\sqrt{3}-\pi/3)$.

3. Let $f = xyz$ and $g = x^2+2y^2+3z^2$. Then $\nabla f = yz\vec{i} + xz\vec{j} + xy\vec{k}$ and $\nabla g = 2x\vec{i} + 4y\vec{j} + 6z\vec{k}$. So, $\nabla f(1,1,1) = \vec{i}+\vec{j}+\vec{k}$ and $\nabla g(1,1,1) = 2\vec{i}+4\vec{j}+6\vec{k}$, which are orthogonal to the level surfaces $f=1$ and $g=6$ respectively. The tangent line is parallel to $\vec{v} = \nabla f \times \nabla g = 2\vec{i} - 4\vec{j} + 2\vec{k} \rightarrow$ Tangent line is given by: $x = 1+2t, y = 1-4t, z = 1+2t$.

4. $\int_0^2\int_0^{2-y}(12-y^2)\,dxdy = \int_0^2(2-y)(12-y^2)dy = 20.$

5. $f_x(x,y) = 4x+3y, f_y(x,y) = 3x-3y^2$. So $f(2,-1) = 3, f_x(2,-1) = 5, f_y(2,-1) = 3$. Therefore, $L(x,y) = 3+5(x-2)+3(y+1)$ and $L(2.1,-1.03) = 3.41$.

6. $\int_0^2\int_0^1\int_{x^2}^1 12xze^{zy^2}\,dy\,dx\,dz = \int_0^2\int_0^1\int_0^{\sqrt{y}} 12xze^{zy^2}\,dx\,dy\,dz = \int_0^2\int_0^1 6yze^{zy^2}\,dy\,dz = \int_0^2[3e^{zy^2}]_0^1\,dz =$ $3\int_0^2(e^z-1)\,dz = 3e^2-9.$

7. For (x,y,z) on the helix, f is a function of t alone. By the chain rule, $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} =$ $(\cos t)(-\sin t) + (\sin t)(\cos t) + t^2 - (\pi+1)t + \pi$. Therefore, $\frac{df}{dt} = 0$ when $t^2-(\pi+1)t+\pi = (t-\pi)(t-1) = 0 \rightarrow t = 1, \pi \rightarrow$ points are $(\cos 1, \sin 1, 1)$ and $(-1, 0, \pi)$.

8. Optimize $f(x,y) = (x-1)^2+(y-2)^2$ subject to the constraint $x^2+y^2 = 45$. Then $\nabla f = \lambda\nabla g \rightarrow 2(x-1) = \lambda 2x$ and $2(y-2) = \lambda 2y \rightarrow \frac{x-1}{x} = \lambda = \frac{y-2}{y} \rightarrow y = 2x$. Therefore, since $x^2+y^2 = 45$ we must have $x^2+4x^2 = 45 \rightarrow x = 3, -3$. So the points are $(3,6)$ and $(-3,-6)$.$f(3,6) = 20$ and $f(-3,-6) = 80 \rightarrow (3,6)$ is at the minimum distance and $(-3,-6)$ is at the maximum distance.