Math 1920, Prelim 2
October 30, 2008
Solutions

1) a) $\nabla f = 2y\mathbf{i} + (2x - 2z)\mathbf{j} - y\mathbf{k}$ so $\nabla f(1,-1,1) = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $|\nabla f(1,-1,1)| = \sqrt{(-2)2 + 12 + 12} = \sqrt{6}$.
Consequently, the direction of most rapid increase is:

$$\mathbf{u}_{\text{incr}} = \frac{\nabla f(1,-1,1)}{|\nabla f(1,-1,1)|} = -\frac{2}{\sqrt{6}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k}$$

and the direction of most rapid decrease is:

$$\mathbf{u}_{\text{decr}} = -\frac{\nabla f(1,-1,1)}{|\nabla f(1,-1,1)|} = \frac{2}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$$

b) The directional derivative in the direction of most rapid increase:

$$D_{\mathbf{u}_{\text{incr}}} f \mid_{(1,-1,1)} = \nabla f \mid_{(1,-1,1)} \cdot \mathbf{u}_{\text{incr}} = \frac{1}{\sqrt{6}}(4 + 1 + 1) = \sqrt{6}$$

The directional derivative in the direction of most rapid decrease:

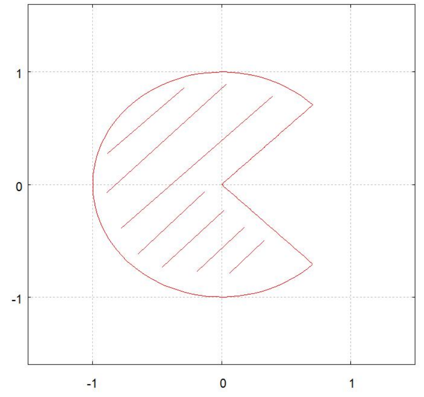
$$D_{\mathbf{u}_{\text{decr}}} f \mid_{(1,-1,1)} = \nabla f \mid_{(1,-1,1)} \cdot \mathbf{u}_{\text{decr}} = -\frac{1}{\sqrt{6}}(4 + 1 + 1) = -\sqrt{6}$$

c) No. The minimum rate of change is $-\sqrt{6} > -3$.

2) a) $f(1,1,2) = -2$, $f_x = z$, $f_x(1,1,2) = 2$, $f_y = -3z$, $f_y(1,1,2) = -6$, $f_z = x - 3y$, $f_z(1,1,2) = -2$ so $L(x,y,z) = -2 + 2(x-1) - 6(y-1) - 2(z-2) = 2x - 6y - 2z + 6$
b) An approximate value of f at $(1.1, 1.1, 2.1)$:

$$L(1.1, 1.1, 2.1) = -2 + 2(1.1 - 1) - 6(1.1 - 1) - 2(2.1 - 2) = -2 + 0.2 - 0.6 - 0.2 = -2.6$$

3) a)



b) We observe that the region is $\frac{3}{4}$ of a unit circle so the area must be $\frac{3}{4} \cdot 1^2 \cdot \pi = \frac{3\pi}{4}$.

Alternatively we can find the area with the following double integral in polar coordinates: $\int_{\pi/4}^{7\pi/4}\int_0^1 r\ dr\ d\theta = \int_{\pi/4}^{7\pi/4} \frac{1}{2}\ d\theta = \frac{1}{2}(\frac{7\pi}{4} - \frac{\pi}{4}) = \frac{3\pi}{4}$.

c) Symmetry around the x-axis gives $\bar{y} = 0$. Next check:

$$\iint_R x\ dA = \int_{\pi/4}^{7\pi/4}\int_0^1 r\cos(\theta) r\ dr\ d\theta = \frac{1}{3}\int_{\pi/4}^{7\pi/4} \cos(\theta)\ d\theta = \frac{1}{3}(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}) = -\frac{\sqrt{2}}{3}$$

Thus we have $\bar{x} = \frac{-\frac{\sqrt{2}}{2}}{\frac{3\pi}{4}} = -\frac{4\sqrt{2}}{9\pi}$

4) The sum of the distances squared from (x, y) are

$$f(x,y) = x^2+y^2+(x-1)^2+y^2+(x-a)^2+(y-b)^2 = 3x^2+3y^2-2(1+a)x-2by+a^2+b^2+1.$$

Then,

$$\nabla f =< 6x - 2(1+a)x, 6y - 2b > .$$

So $\nabla f =< 0, 0 >$, only when $x = (1 + a)/3$, and $y = b/3$, and so $((1 + a)/3, b/3)$ is the only critical point.

The function f is differentiable at all (x, y), and it is continuous. Since f is the sum of squares, $f(x, y) \geq 0$ for all (x, y), it is bounded below. So if f has a (global) minimum point it must be $((1 + a)/3, b/3)$. But for any (x, y) outside or on the boundary of a large disk D centered at $(0, 0)$, say, f is larger than $f((1 + a)/3, b/3)$. So the minimum of f must be in the interior of D, since D is closed and bounded (compact), and it must be the global minimum of f. So that minimum must be achieved at the critical point $((1 + a)/3, b/3)$.

Note that if you calculated the Hessian correctly, you showed that the critical point is a *local* minimum. For full credit you needed to show this was a global minimum.

5) Assume $x, y, z \geq 0$, for the coordinates in the first octant, and the vertices of the box lie on the surface of the ellipsoid. Then the volume of the box is $V = 8xyz$. Let $f(x, y, z) = x^2 + 2y^2 + 3z^2$. $\nabla V = 8 < yz, xz, xy >$, and $\nabla f =< 2x, 4y, 6z >$. Since V is continuous and differentiable, and the boundary of the ellipsoid is closed and bounded, f must have a maximum, where $\lambda\nabla f = \nabla V$, for some scalar λ, the Lagrange multiplier. So

$$\lambda 2x = 8yz,\ \ \lambda 4y = 8xz,\ \ \lambda 6z = 8xy.$$

Since $V = 0$ when either $x = 0$, or $y = 0$, or $z = 0$, and the maximum is clearly not 0, we can assume that they are all non-zero. So λ is non-zero as well. Dividing the first equation by the second we get $\frac{\lambda 2x}{\lambda 4y} = \frac{8yz}{8xz}$, which simplifies to $x^2 = 2y^2$. Similarly dividing the third equation by the first we get $\frac{\lambda 2x}{\lambda 6z} = \frac{8yz}{8xy}$, which simplifies to $x^2 = 3z^2$. Substituting into the ellipsoid constraint we get $x^2 + x^2 + x^2 = 1$, giving $x = \frac{1}{\sqrt{3}}$. Then $y = \frac{1}{\sqrt{2}}x = \frac{1}{\sqrt{6}}$, and $z = \frac{1}{\sqrt{3}}x = \frac{1}{3}$. So the maximum volume is $8\frac{1}{\sqrt{3}}\frac{1}{\sqrt{6}}\frac{1}{3} = \frac{8}{9\sqrt{2}}$. Note that this is a maximum because the domain for f, the ellipsoid, is closed and bounded so we know that f attains both a global maximum and a global minimum. We can make f as close to 0 as we want by making one of the variables tend to 0, so the value we found must be a maximum.