Math 192, Prelim 2 - Solutions
October 25, 2007. 7:30-9:00

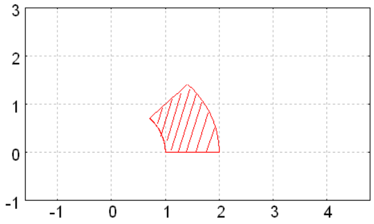
1) a) $3 = 1^2 + 2 \cdot 0^2 + 2$ and $3 = 5 \cdot 1 - 0 - 2$ so $(1,0,3)$ is on both of those surfaces.
b) Set $f(x,y,z) = x^2 + 2y^2 + 2 - z$. Then $f(x,y,z) = 0$ is a level surface. Now $\nabla f(x,y,z) = 2x\mathbf{i} + 4y\mathbf{j} - \mathbf{k}$ and $\nabla f(1,0,3) = 2\mathbf{i} - \mathbf{k}$ is a normal to the tangent plane. Thus the equation of the tangent plane is $2(x-1) + 0(y-0) - (z-3) = 0 \Rightarrow 2x - z = -1$
c) Set $g(x,y,z) = 5x - y - 2 - z$. Then $g(x,y,z) = 0$ is a level surface. Observe that $\nabla g(x,y,z) = 5\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\nabla g(1,0,3) = 5\mathbf{i} - \mathbf{j} - \mathbf{k}$ is a normal to the tangent plane of S_2 at $(1,0,3)$. Since the line tangent to the curve of intersection of S_1 and S_2 is in the tangent planes of both surfaces at $(1,0,3)$ then $\nabla f(1,0,3) \times \nabla g(1,0,3) = -\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$ is a vector parallel to the line. Thus the parametric equation for the line is $x = 1 - t$, $y = -3t$ and $z = 3 - 2t$.

2) a) The domain is $\{(x,y) \in \mathbb{R}^2 | y > 0\}$, that is the domain is the upper half plane, not including the x-axis.
b) $\nabla f(x,y,z) = (\ln(y) + ye^x)\mathbf{i} + (\frac{x}{y} + e^x)\mathbf{j}$, $\nabla f(0,e) = (1+e)\mathbf{i} + \mathbf{j}$, $|\nabla f(0,e)| = \sqrt{(1+e)^2 + 1^2}$ so $\mathbf{u} = \frac{\nabla f(0,e)}{|\nabla f(0,e)|} = \frac{(1+e)\mathbf{i}+\mathbf{j}}{\sqrt{(1+e)^2+1^2}}$ is a unit vector describing the direction of steepest ascent of f at the point $(0,e)$.
c) $D_{\mathbf{u}}f(0,e) = \nabla f(0,e) \cdot \mathbf{u} = \nabla f(0,e) \cdot \frac{\nabla f(0,e)}{|\nabla f(0,e)|} = |\nabla f(0,e)| = \sqrt{(1+e)^2 + 1^2}$
d) Let $\mathbf{v} = \langle v_1, v_2 \rangle$ be such a vector. Then $0 = \mathbf{v} \cdot \nabla f(0,e) = v_1(1+e) + v_2$. If $v_1 = 0$ then that equation shows that $v_2 = 0$ which is a contradiction because $\mathbf{v}$ is a unit vector. If $0 = \mathbf{v} \cdot \nabla f(0,e)$ then we also have that $\frac{1}{v_1}\mathbf{v} = \left\langle 1, \frac{v_2}{v_1} \right\rangle$ is perpendicular to $\nabla f(0,e)$ and if we set $x = \frac{v_2}{v_1}$ then we know that $0 = \langle 1, x \rangle \cdot \nabla f(0,e) = 1 + e + x \Rightarrow x = -1 - e$ which shows that the unit vectors perpendicular to $\nabla f(0,e)$, that is the unit vectors of zero change of f at $(0,e)$, are $\mathbf{v} = \pm \frac{\mathbf{i} - (1+e)\mathbf{j}}{\sqrt{1+(1+e)^2}}$

3) $\nabla f = (4x^3 - 4y)\mathbf{i} + (4y^3 - 4x)\mathbf{j}$ so $4x^3 - 4y = 0$ and $4y^3 - 4x = 0 \Rightarrow x^3 = y$ and $x = y^3 \Rightarrow y^9 = y \Rightarrow y = 0$, -1 or 1. Thus critical points are $(0,0)$, $(-1,-1)$ and $(1,1)$. Now check that $f_{xx}(x,y) = 12x^2$, $f_{yy}(x,y) = 12y^2$, $f_{xy}(x,y) = -4$. Thus we have $f_{xx}(0,0)f_{yy}(0,0) - f^2_{xx}(0,0) = -16 < 0$ so $(0,0)$ is a saddle point, $f_{xx}(-1,-1)f_{yy}(-1,-1) - f^2_{xx}(-1,-1) = 12 \cdot 12 - 16 > 0$ and $f_{xx}(-1,-1) = 12 > 0$ so $f(-1,-1) = -1$ is a local minimum and finally $f_{xx}(1,1)f_{yy}(1,1) - f^2_{xx}(1,1) = (12) \cdot (12) - 16 > 0$ and $f_{xx}(1,1) = 12 > 0$ so $f(1,1) = -1$ is a local minimum.

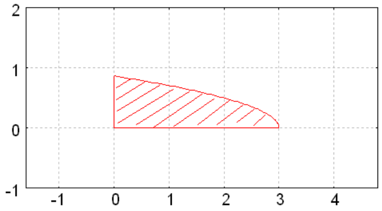
4) Set $f(x,y,z) = (x-2)^2 + (y-1)^2 + (z-3)^2$ and let's find minima and maxima of f subject to $g(x,y,z) = 4$, where $g(x,y,z) = x^2 + y^2 + z^2$, by using the method of Lagrange multipliers. Now $\nabla f = 2(x-2)\mathbf{i} + 2(y-1)\mathbf{j} + 2(z-3)\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so $\nabla f = \lambda \nabla g \Rightarrow 2(x-2) = 2x\lambda$, $2(y-1) = 2y\lambda$ and $2(z-3) = 2z\lambda \Rightarrow 2x(1-\lambda) = 4$, $2y(1-\lambda) = 2$, $2z(1-\lambda) = 6$. Observe that if $\lambda = 1$ then those equations would show $0 = 4$, $0 = 2$ and $0 = 6$ which clearly would be a contradiction. Thus $\lambda \neq 1$ and we get $x = \frac{2}{1-\lambda}$, $y = \frac{1}{1-\lambda}$ and $z = \frac{3}{1-\lambda}$. Use this with the equation $g(x,y,z) = 4$ and get $\frac{4}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{9}{(1-\lambda)^2} = 4 \Rightarrow 14 = 4(1-\lambda)^2 \Rightarrow 1 - \lambda = \pm\sqrt{\frac{7}{2}} \Rightarrow \lambda = 1 \pm \sqrt{\frac{7}{2}}$. Thus we have two solutions, $(2\sqrt{\frac{2}{7}}, \sqrt{\frac{2}{7}}, 3\sqrt{\frac{2}{7}})$ and $(-2\sqrt{\frac{2}{7}}, -\sqrt{\frac{2}{7}}, -3\sqrt{\frac{2}{7}})$. If we compute the distance of those two solution to the point $(2,1,3)$ then we see that $(2\sqrt{\frac{2}{7}}, \sqrt{\frac{2}{7}}, 3\sqrt{\frac{2}{7}})$ is closest to $(2,1,3)$ and $(-2\sqrt{\frac{2}{7}}, -\sqrt{\frac{2}{7}}, -3\sqrt{\frac{2}{7}})$ is farthest from $(2,1,3)$.

5) a)



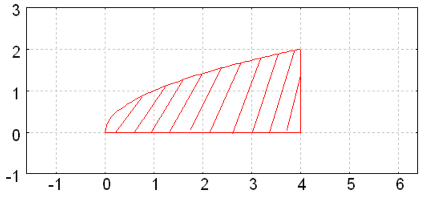
b)$\int\int(1+x+xy)dxdy = \int_0^{\pi/4}\int_1^2(1+r\cos\theta+r^2\cos\theta\sin\theta)rdrd\theta = \int_0^{\pi/4}\left[\frac{r^2}{2}+\frac{r^3}{3}\cos\theta+\frac{r^4}{8}\sin(2\theta)\right]_1^2 d\theta =$ $\int_0^{\pi/4}(\frac{3}{2}+\frac{7}{3}\cos\theta+\frac{15}{8}\sin(2\theta))d\theta = \frac{3\pi}{8}+\left[\frac{7}{3}\sin\theta\right]_0^{\pi/4}+\left[-\frac{15}{16}\cos(2\theta)\right]_0^{\pi/4} = \frac{3\pi}{8}+\frac{7\sqrt{2}}{6}+\frac{15}{16}$

6) a)



b)$\int\int xy^3dxdy = \int_0^{\sqrt{3}/2}\int_0^{3-4y^2}xy^3dxdy = \int_0^{\sqrt{3}/2}\frac{1}{2}(3-4y^2)^2y^3dy = \int_0^{\sqrt{3}/2}(\frac{9}{2}y^3-12y^5+8y^7)dy =$ $\left[\frac{9}{8}y^4-2y^6+y^8\right]_0^{\sqrt{3}/2} = \frac{81}{128}-\frac{27}{32}+\frac{81}{256} = \frac{27}{256}$

7) a)



b) $\int_0^2\int_{y^2}^4\frac{y^3}{x}e^{x^2}dxdy = \int_0^4\int_0^{\sqrt{x}}\frac{y^3}{x}e^{x^2}dydx = \int_0^4\frac{\frac{1}{4}(\sqrt{x})^4}{x}e^{x^2}dx = \frac{1}{4}\int_0^4 xe^{x^2}dx = \frac{1}{4}\left[\frac{1}{2}e^{x^2}\right]_0^4 = \frac{1}{8}e^{16}-\frac{1}{8}$