MATH 1920 SUMMER 2009 PRELIM 2

SOLUTIONS

1. (8 pts) Consider the function $f(x,y) = \ln xy + \ln yz + \ln xz$.

   (a) Find the directions in which $f(x,y,z)$ increases and decreases most rapidly at $P_0(1,1,1)$.
   Solution. $f(x,y,z)$ increases most rapidly at $P_0(1,1,1)$ in the direction of $\nabla f(1,1,1)$ and decreases most rapidly at $P_0(1,1,1)$ in the direction of $-\nabla f(1,1,1)$. We have

$$\frac{\partial f}{\partial x}(1,1,1) = \left(\frac{1}{x} + 0 + \frac{1}{x}\right)\Big|_{(x,y,z)=(1,1,1)} = 2,$$
$$\frac{\partial f}{\partial y}(1,1,1) = \left(\frac{1}{y} + \frac{1}{y} + 0\right)\Big|_{(x,y,z)=(1,1,1)} = 2,$$
$$\frac{\partial f}{\partial z}(1,1,1) = \left(0 + \frac{1}{z} + \frac{1}{z}\right)\Big|_{(x,y,z)=(1,1,1)} = 2,$$

   Hence $\nabla f(1,1,1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $-\nabla f(1,1,1) = -2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$.

   (b) Find the derivatives of $f(x,y,z)$ in these directions.

   Solution. Consider the unit vector

$$\mathbf{u} = \frac{\nabla f(1,1,1)}{|\nabla f(1,1,1)|} = \frac{2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}{|2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}|} = \frac{2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}{2\sqrt{3}} = \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}$$

   The derivative of f in the direction of $\nabla f(1,1,1)$ is

$$D_{\mathbf{u}} f(1,1,1) = \nabla f(1,1,1) \cdot \mathbf{u} = |\nabla f(1,1,1)| = 2\sqrt{3},$$

   while the derivative of f in the direction of $-\nabla f(1,1,1)$ is

$$D_{-\mathbf{u}} f(1,1,1) = \nabla f(1,1,1) \cdot (-\mathbf{u}) = -|\nabla f(1,1,1)| = -2\sqrt{3}.$$

2. (7 pts) Find parametric equations for the line tangent to the curve of intersection of the surfaces $xyz = 1$ and $x^2 + 2y^2 + 3z^2 = 6$ at the point $(1,1,1)$.
   Solution. Let $f(x,y,z) = xyz$ and $g(x,y,z) = x^2 + 2y^2 + 3z^2$. The tangent line in question is orthogonal to both ∇f and ∇g at the point $(1,1,1)$, and therefore parallel to $\mathbf{v} = \nabla f \times \nabla g$. We have

$$\nabla f\Big|_{(1,1,1)} = (yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k})\Big|_{(1,1,1)} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$
$$\nabla g\Big|_{(1,1,1)} = (2x\mathbf{i} + 4y\mathbf{j} + 6z\mathbf{k})\Big|_{(1,1,1)} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$$
$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 4 & 6 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}.$$

The tangent line is

$$x = 1 + 2t, \quad y = 1 - 4t, \quad z = 1 + 2t, \quad -\infty < t < \infty.$$

3. (8 pts) Find all critical points of the function $f(x, y) = x^2y + xy^2 - 3xy$ and determine their nature. Justify your answer.
Solution. We have

$$\frac{\partial f(x,y)}{\partial x} = 2xy + y^2 - 3y \quad \text{and} \quad \frac{\partial f(x,y)}{\partial y} = x^2 + 2xy - 3x,$$

for all $(x, y) \in \mathbb{R}^2$. Since the partial derivatives exist everywhere, in order to find the critical points of f we need only solve the system

$$2xy + y^2 - 3y = 0 \quad \text{and} \quad x^2 + 2xy - 3x = 0,$$

which is equivalent to the system

$$\begin{aligned} y(2x + y - 3) &= 0 \\ x(x + 2y - 3) &= 0 \end{aligned}$$

The first equation is true if and only if $y = 0$ or $2x + y = 3$, while the second equation holds if and only if $x = 0$ or $x + 2y = 3$.

So one way to solve this system is to think geometrically: consider the four lines

$$\begin{aligned} L1: &\quad y = 0 \\ L2: &\quad 2x + y = 3 \\ L3: &\quad x = 0 \\ L4: &\quad x + 2y = 3 \end{aligned}$$

and note that the solutions of the above system are the points of intersection of the four pairs of lines $L1$ and $L3$, $L1$ and $L4$, $L2$ and $L3$, and $L2$ and $L4$, which are $(0, 0)$, $(0, 3)$, $(3, 0)$ and $(1, 1)$, respectively.

The Hessian of f is equal to

$$H(x,y) = \begin{vmatrix} \dfrac{\partial^2 f(x,y)}{\partial x^2} & \dfrac{\partial^2 f(x,y)}{\partial x \partial y} \\ \dfrac{\partial^2 f(x,y)}{\partial y \partial x} & \dfrac{\partial^2 f(x,y)}{\partial x^2} \end{vmatrix} = \begin{vmatrix} 2y & 2x + 2y - 3 \\ 2x + 2y - 3 & 2x \end{vmatrix} = 4xy - (2x + 2y - 3)^2,$$

for all $(x, y) \in \mathbb{R}^2$. Therefore, $(0, 0)$, $(0, 3)$, and $(3, 0)$ are **saddle points**, because

$$H(0, 0) = H(3, 0) = H(0, 3) = -9 < 0.$$

Since $H(1, 1) = 3 > 0$ and

$$\frac{\partial^2 f(1,1)}{\partial x^2} = 2y\Big|_{(x,y)=(1,1)} = 2 > 0,$$

f has a **local minimum** at $(1, 1)$.

4. (7 pts) Find the points on the surface $z^2 = xy + 4$ closest to the origin.

First Solution. It suffices to find the minimum of the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = 4$, where $g(x, y, z) = z^2 - xy$. To this end, we find the values of x, y, z and λ that simultaneously satisfy the equations

$$\begin{aligned} \nabla f(x, y, z) &= \lambda \nabla g(x, y, z), \\ g(x, y, z) &= 4, \end{aligned}$$

Since

$$\nabla f(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k},$$

and

$$\nabla g(x, y, z) = -y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k},$$

we must solve the following system

$$\begin{aligned} 2x &= -\lambda y, && (1) \\ 2y &= -\lambda x, && (2) \\ 2z &= 2\lambda z, && (3) \\ z^2 - xy &= 4. && (4) \end{aligned}$$

We see that (3) is equivalent to $z(1 - \lambda) = 0$, which is true if and only if $z = 0$ or $\lambda = 1$. We examine the following two cases

Case 1. $z = 0$.

In this case, $xy = -4$ from (4). Multiplying (1) and (2) gives us $4xy = \lambda^2 xy$, or equivalently $4 = \lambda^2$ because $xy \neq 0$, and so $\lambda = \pm 2$. If $\lambda = -2$, then (1) implies that $x = y$, which contradicts $xy = -4$. Hence $\lambda = 2$ and $x = -y$ (from (1)), which combined with $xy = -4$ gives us the two pairs $(x, y) = (2, -2)$ and $(x, y) = (-2, 2)$.

Case 2. $z \neq 0$.

In this case $\lambda = 1$. Then the system of (1) and (2) has only one solution $(x, y) = (0, 0)$, and so $z = \pm 2$ from (4).

Therefore we get four points $(2, -2, 0)$, $(-2, 2, 0)$, $(0, 0, 2)$ and $(0, 0, -2)$. The points $(0, 0, 2)$ and $(0, 0, -2)$ are closest to the origin because they are 2 units away, while the others are $2\sqrt{2}$ units away.

Second Solution. It suffices to find the minimum of the function $f(x, y) = x^2 + y^2 + xy + 4$. This is essentially problem (3) of the Summer 2007 second preliminary exam. Repeating that solution we see that the absolute minimum of f occurs at $(x, y) = (0, 0)$. Then $z^2 = xy + 4 = 4$, forcing $z = \pm 2$. Therefore $(0, 0, 2)$ and $(0, 0, -2)$ are the points on $z^2 = xy + 4$ closest to the origin.

5. (8 pts) Find the volume of the solid that is bounded above by the surface $z = x^2 e^{xy}$ and below by the triangular region R in the xy-plane enclosed by the lines $y = x$, $y = 0$, and $x = 1$.
Solution. The volume in question equals the double integral

$$\iint\limits_R x^2 e^{xy}\, dA,$$

where the region of integration R is

$$R = \{(x, y) |\, 0 \le x \le 1, \quad 0 \le y \le x\}.$$

Hence the volume equals

$$\begin{aligned}\int_0^1 \int_0^x x^2 e^{xy}\, dy\, dx &= \int_0^1 \Big[xe^{xy}\Big]_{y=0}^{y=x} dx\\ &= \int_0^1 (xe^{x^2} - x)\, dx\\ &= \left[\frac{e^{x^2} - x^2}{2}\right]_{x=0}^{x=1}\\ &= \boxed{\frac{e-2}{2}} \quad \text{units cubed.}\end{aligned}$$

6. (7 pts) Evaluate the double integral

$$\int_0^2 \int_0^{\sqrt{1-(x-1)^2}} \frac{y}{x^2 + y^2}\, dy\, dx.$$

Solution. We first note that the iterated integral in question equals the double integral

$$\iint\limits_R \frac{y}{x^2 + y^2}\, dy\, dx,$$

where R is the semicircular region bounded by the curve

$$y = \sqrt{1 - (x-1)^2} = \sqrt{2x - x^2}, \quad (0 \le x \le 2)$$

and the x-axis. Substituting $x = r\cos\theta$ and $y = r\sin\theta$ into

$$y^2 = 2x - x^2,$$

and since $x^2 + y^2 = r^2$, we see that the polar form of the graph of

$$y = \sqrt{2x - x^2}, \quad 0 \le x \le 2$$

is

$$r = 2\cos\theta, \quad 0 \le \theta \le \pi/2.$$

The polar form of R is

$$R = \{(r,\theta) |\, 0 \le r \le 2\cos\theta, \quad 0 \le \theta \le \pi/2\}.$$

In polar coordinates, the given integral becomes

$$\begin{aligned}\int_0^{\pi/2}\int_0^{2\cos\theta} \frac{r\sin\theta}{r^2}\, r dr\, d\theta &= \int_0^{\pi/2}\int_0^{2\cos\theta} \sin\theta\, dr\, d\theta \\ &= \int_0^{\pi/2} 2\cos\theta\sin\theta\, d\theta \\ &= \Big[\sin^2\theta\Big]_{\theta=0}^{\theta=\pi/2} \\ &= \boxed{1}.\end{aligned}$$

7. (9 pts) Let D be the region in the *first octant* that is bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the sphere $x^2+y^2+z^2 = 9$.

   (a) Express the volume of D as an iterated triple integral in

      (i) spherical,
      Solution. The equations of the cone and the sphere in spherical coordinates are $\phi = \pi/4$ and $\rho = 3$, respectively. In spherical coordinates,

$$D = \{(\rho,\phi,\theta) \,|\, 0 \le \rho \le 3, 0 \le \phi \le \pi/4, 0 \le \theta \le \pi/2\}.$$

      and so

$$V(D) = \int_0^{\pi/2}\int_0^{\pi/4}\int_0^3 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

      (ii) cylindrical,
      Solution. The equations of the cone and the sphere in cylindrical coordinates are $z = r$ and $r^2 + z^2 = 9$, respectively. Their curve of intersection is on the plane $z = \sqrt{\frac{9}{2}}$, and so, in cylindrical coordinates,

$$D = \{(r,\theta,z) \,|\, 0 \le r \le \sqrt{\frac{9}{2}},\, 0 \le \theta \le \pi/2,\, r \le z \le \sqrt{9-r^2}\}.$$

      and so

$$V(D) = \int_0^{\pi/2}\int_0^{\sqrt{9/2}}\int_r^{\sqrt{9-r^2}} dz\, r dr\, d\theta$$

      (iii) rectangular coordinates.
      Solution. In rectangular coordinates,

$$D = \{(x,y,z) \,|\, 0 \le x \le \sqrt{\frac{9}{2}},\, 0 \le y \le \sqrt{\frac{9}{2} - x^2},\, \sqrt{x^2+y^2} \le z \le \sqrt{9-x^2-y^2}\}.$$

and so

$$V(D) = \int_0^{\sqrt{\frac{9}{2}}} \int_0^{\sqrt{\frac{9}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{9-x^2-y^2}} dz\, dy\, dx$$

(b) Find the volume of D.
Solution. In spherical coordinates,

$$\begin{aligned} V(D) &= \int_0^{\pi/2} \int_0^{\pi/4} \int_0^3 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\ &= 9 \int_0^{\pi/2} \int_0^{\pi/4} \sin\phi\, d\phi\, d\theta \\ &= 9 \int_0^{\pi/2} \left(1 - \frac{1}{\sqrt{2}}\right) d\theta \\ &= \frac{9\pi}{2}\left(1 - \frac{1}{\sqrt{2}}\right). \end{aligned}$$

Remark. In cylindrical coordinates, the computation of $V(D)$ is a little more difficult, but still doable.

$$\begin{aligned} V(D) &= \int_0^{\pi/2} \int_0^{\sqrt{9/2}} (r\sqrt{9-r^2} - r^2)\, dr\, d\theta \\ &= \int_0^{\pi/2} \left[-\frac{(9-r^2)^{3/2} + r^3}{3} \right]_{r=0}^{r=\sqrt{9/2}} d\theta \\ &= \frac{\pi}{2} \cdot \frac{9^{3/2} - 2\left(\frac{9}{2}\right)^{3/2}}{3} \\ &= \frac{\pi}{2} \cdot \frac{27 - \frac{27}{\sqrt{2}}}{3} \\ &= \frac{9\pi}{2}\left(1 - \frac{1}{\sqrt{2}}\right). \end{aligned}$$

In rectangular coordinates the computation of $V(D)$ is a "bit" messier.

8. (6 pts) Evaluate $\int_C (x+y+z)\, ds$, where C is the straight line segment from $(1,2,3)$ to $(0,-1,1)$.
Solution. Consider the continuous function $f(x,y,z) = x+y+z$ over C. The given straight line segment passes through the point $(1,2,3)$ and is parallel to the vector $-\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$. The parametric equations of C are

$$x = 1-t, \quad y = 2-3t, \quad z = 3-2t, \quad 0 \le t \le 1.$$

Thus, a parametrization of C is

$$\mathbf{r}(t) = (1-t)\,\mathbf{i} + (2-3t)\,\mathbf{j} + (3-2t)\mathbf{k}, \quad 0 \le t \le 1,$$

which is smooth since $|\mathbf{r}'(t)| = |-\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}| = \sqrt{14} \neq 0$ for all $0 \le t \le 1$. Therefore

$$\begin{aligned}
\int_C (x+y+z)\,ds &= \int_C f(1-t, 2-3t, 3-2t)|\mathbf{r}'(t)|\,dt \\
&= \int_0^1 6(1-t)\sqrt{14}\,dt \\
&= 6\sqrt{14}\Big[t - \frac{t^2}{2}\Big]_{t=0}^{t=1} \\
&= 6\sqrt{14} \cdot (\frac{1}{2} - 0) \\
&= \boxed{3\sqrt{14}}.
\end{aligned}$$