Math 1920, Prelim 1
October 2, 2008
Solutions

1) a) $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ is a normal of the plane and $|\mathbf{n}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$. By looking at the equation of the plane then we see that $P = (1, 1, -1)$ is a point in the plane. If we set $S = (1, 1, 1)$ then we have $\overrightarrow{PS} =< 1 - 1, 1 - 1, 1 - (-1) >=< 0, 0, 2 >$ and the distance from $(1, 1, 1)$ to the plane is $d = |\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}| = |\frac{0\cdot1+0\cdot2+2\cdot3}{\sqrt{14}}| = \frac{6}{\sqrt{14}}$.
b) $< 1 - 0, 1 - 0, 0 - 0 >=< 1, 1, 0 >$ and $< 1 - 0, 2 - 0, 3 - 0 >=< 1, 2, 3 >$ are two vectors in the plane that are not parallel. A vector perpendicular to the plane is then given by the cross product between the two vectors.

$$< 1, 1, 0 > \times < 1, 2, 3 >= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & 2 & 3 \end{vmatrix} = (3 - 0)\mathbf{i} - (3 - 0)\mathbf{j} + (2 - 1)\mathbf{k} = 3\mathbf{i} - 3\mathbf{j} + \mathbf{k}$$

So the equation is given by $3x - 3y + z = 0$, since the plane goes through the origin $(0, 0, 0)$.

2) The velocity vector is $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = -\sin(t)\mathbf{i} + 2\mathbf{j} + \cos(t)\mathbf{k}$. The acceleration vector is $\mathbf{a}(t) = \frac{d\mathbf{r}}{dt} = -\cos(t)\mathbf{i} - \sin(t)\mathbf{k}$. To calculate the angle between $\mathbf{v}(t)$ and $\mathbf{a}(t)$ consider the dot product between these two vectors:

$$\mathbf{v}(t) \cdot \mathbf{a}(t) = \cos(t)\sin(t) - \cos(t)\sin(t) = 0$$

The product is always 0. Hence, the velocity and acceleration vectors are always orthogonal to each other.

3) Differentiating both sides of equation w.r.t. z, we get

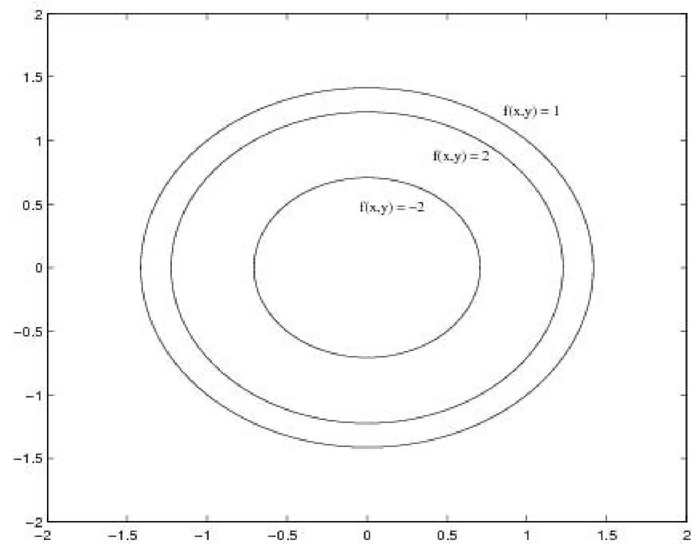
$$3y \cdot 2x \cdot \frac{\partial x}{\partial z} + y + \ln(2x - 1) + z \cdot \frac{2}{2x - 1} \cdot \frac{\partial x}{\partial z} = 0$$

$$(6xy + \frac{2z}{2x - 1})\frac{\partial x}{\partial z} = -y - \ln(2x - 1)$$

So at point $(1, -1, -3)$ we get $\left((6)(1)(-1) + \frac{2(-3)}{2(1)-1}\right)\frac{\partial x}{\partial z} = -(-1) - \ln(2(1) - 1)$ and thus $\frac{\partial x}{\partial z} = \frac{1-0}{-6+(-6)} = -\frac{1}{12}$.

4) a) The domain is the set of all points (x, y) in the plane that fulfill the equation $x^2+y^2 \neq 1$. In other words it is the set of all points in the plane except for the unit circle.
b) $f(x, y) = -2$ or $\frac{1}{x^2+y^2-1} = -2$ gives $x^2 + y^2 = \frac{1}{2}$ which is a circle centered at $(0, 0)$ with radius $\frac{1}{\sqrt{2}}$.
$f(x, y) = 1$ or $\frac{1}{x^2+y^2-1} = 1$ gives $x^2 + y^2 = 2$ which is a circle centered at $(0, 0)$ with radius $\sqrt{2}$.
$f(x, y) = 2$ or $\frac{1}{x^2+y^2-1} = 2$ gives $x^2 + y^2 = \frac{3}{2}$ which is a circle centered at $(0, 0)$ with radius $\sqrt{\frac{3}{2}}$.
c) $f(x, y) = -1$ or $\frac{1}{x^2+y^2-1} = -1$ gives $x^2 + y^2 = 0$ so the level curve is the point $(0, 0)$.
$f(x, y) = -\frac{1}{2}$ or $\frac{1}{x^2+y^2-1} = -\frac{1}{2}$ gives $x^2 + y^2 = -1$ which clearly can not hold for any point

(x, y). Thus the level curve does not exist.
$f(x, y) = 0$ or $\frac{1}{x^2+y^2-1} = 0$ gives $1 = 0$ and can thus clearly not hold for any point (x, y). Hence the level curve does not exist.



5) a) $\mathbf{v}(t) = 4\cos(t)\mathbf{i} - 4\sin(t)\mathbf{j} + 3\mathbf{k}$ and $|\mathbf{v}(t)| = \sqrt{(4\cos(t))^2 + (-4\sin(t))^2 + 3^2} = \sqrt{4^2+3^2} = 5$. The unit tangent vector is $\mathbf{T} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{4}{5}\cos(t)\mathbf{i} - \frac{4}{5}\sin(t)\mathbf{j} + \frac{3}{5}\mathbf{k}$.
b) Let t be the time when we are the point we want to find. Then we must have $10\pi = \int_0^t |\mathbf{v}(\tau)|d\tau = 5t$ and thus $t = 2\pi$. Since $\mathbf{r}(2\pi) =< 0, 4, 6\pi >$ we see that the point is $(0, 4, 6\pi)$.

6) $\overrightarrow{AB} = (0-1)\mathbf{i} + (1-0)\mathbf{j} + (1-0)\mathbf{k} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ is a vector parallel to the line through A and B. Using that vector and the point $A = (1, 0, 0)$ then a parametrization of the line segment from A to B is $\mathbf{r}(t) = (1-t)\mathbf{i} + t\mathbf{j} + t\mathbf{k}$ where $0 \le t \le 1$. Next we see that $\mathbf{v}(t) = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $|\mathbf{v}(t)| = \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}$. Finally we calculate the line integral,

$$\begin{aligned}
\int_C f(x, y, z)ds &= \int_0^1 \left((1-t)^2 + 2t^2 + 3t^2\right)\sqrt{3}dt \\
&= \int_0^1 (1 - 2t + 6t^2)\sqrt{3}dt \\
&= \sqrt{3}\left[t - t^2 + 2t^3\right]_0^1 \\
&= 2\sqrt{3}
\end{aligned}$$

7) Since the angle between any different pair of the vectors $\mathbf{v}_i$ and $\mathbf{v}_j$ is θ, and they are unit vectors we have $\mathbf{v}_i \cdot \mathbf{v}_j = \cos\theta$, for $i \neq j$. Then

$$0 = \mathbf{v}_1 \cdot (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4) = \mathbf{v}_1 \cdot \mathbf{v}_1 + \mathbf{v}_1 \cdot \mathbf{v}_2 + \mathbf{v}_1 \cdot \mathbf{v}_3 + \mathbf{v}_1 \cdot \mathbf{v}_4 = 1 + 3\cos\theta.$$

So $\theta = \cos^{-1}(-1/3)$.