MATH 1920 SUMMER 2009 PRELIM 1

SOLUTIONS

1. (7 pts) The position vector of a moving particle in space is given by

$$\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}, \quad 0 \le t \le 2\pi.$$

Find the time or times in the given time interval when the velocity and acceleration vectors are orthogonal.
Solution. The velocity vector is

$$\mathbf{v}(t) = \mathbf{r}'(t) = (1 - \cos t)\mathbf{i} + \sin t\mathbf{j} \quad (0 \le t \le 2\pi),$$

and the acceleration vector is

$$\mathbf{a}(t) = \mathbf{v}'(t) = \sin t\mathbf{i} + \cos t\mathbf{j} \quad (0 \le t \le 2\pi)$$

Since

$$\mathbf{v}(t) \cdot \mathbf{a}(t) = (1 - \cos t)\sin t + \sin t \cos t = \sin t,$$

the velocity and acceleration vectors are orthogonal when $t = 0, \pi$, or 2π.

2. (7 pts) The temperature at a point (x, y) on a flat metal plate is given by

$$T(x, y) = \frac{60}{1 + x^2 + y^2},$$

where T is measured in $^\circ C$ and x, y in meters. Find the rate of change of temperature with respect to distance at the point $(2, 1)$ in

(a) the **i**-direction.
Solution. The rate of change of temperature with respect to distance at the point $(2, 1)$ in the **i**-direction is equal to

$$\frac{\partial T}{\partial x}(2,1) = \frac{\partial}{\partial x}\left(\frac{60}{1 + x^2 + y^2}\right)\Big|_{(x,y)=(2,1)} = \frac{-120x}{(1 + x^2 + y^2)^2}\Big|_{(x,y)=(2,1)} = -\frac{20}{3}\,^\circ C/m.$$

(b) the **j**-direction.
Solution. The rate of change of temperature with respect to distance at the point $(2, 1)$ in the **j**-direction is equal to

$$\frac{\partial T}{\partial y}(2,1) = \frac{\partial}{\partial y}\left(\frac{60}{1 + x^2 + y^2}\right)\Big|_{(x,y)=(2,1)} = \frac{-120y}{(1 + x^2 + y^2)^2}\Big|_{(x,y)=(2,1)} = -\frac{10}{3}\,^\circ C/m.$$

3. (10 pts) Find the following limits, if they exist.

(a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$.
Solution. Direct substitution gives us $\frac{0}{0}$, and so it does not apply. Note that if $y = \lambda x$, $x \neq 0$, then

$$\frac{xy}{x^2+y^2} = \frac{x(\lambda x)}{x^2+\lambda^2 x^2} = \frac{\lambda}{1+\lambda^2}$$

If (x, y) approaches the origin along the line $y = 0$ (i.e. $\lambda = 0$), then

$$\frac{xy}{x^2+y^2} = \frac{\lambda}{1+\lambda^2} = 0$$

approaches 0, while if (x, y) approaches the origin along the line $y = x$ (i.e. $\lambda = 1$), then

$$\frac{xy}{x^2+y^2} = \frac{\lambda}{1+\lambda^2} = \frac{1}{2}$$

approaches $1/2 \neq 0$. By the two-path test, the limit in question does not exist.

(b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^2}$.
Solution. Direct substitution gives us $\frac{0}{0}$, and so it does not apply. For all $(x, y) \neq (0, 0)$, we have

$$0 \leq \frac{x^2y^2}{x^2+y^2} \leq x^2.$$

Since $\lim\limits_{(x,y)\to(0,0)} x^2 = \lim\limits_{(x,y)\to(0,0)} 0 = 0$, the sandwich theorem implies that

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0.$$

4. Consider the function $f(x, y) = \dfrac{1}{\sqrt{9-x^2-y^2}}$.

(a) (6 pts) Find the domain D and the range R of f.
Solution. The domain of f is clearly the set

$$D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 9\}.$$

For each $(x, y) \in D$, we have $0 < 9 - (x^2 + y^2) \leq 9$, and so

$$0 < \sqrt{9-(x^2+y^2)} \leq \sqrt{9} = 3.$$

Hence

$$f(x, y) = \frac{1}{\sqrt{9-x^2-y^2}} \geq \frac{1}{3}.$$

Therefore the range of f is $R = [\frac{1}{3}, \infty)$.

(b) (2 pts) Determine if the domain of f is an open region, a closed region, or neither.
Solution. The domain D of f is an open region because each point of D is an interior point.

(c) (3pts) Describe and graph the function's level curves $\{(x, y) \in D : f(x, y) = c\}$ for $c = 1/3$, $c = 1/2$ and $c = 1$.
Solution. The level curve $\{(x, y) \in D : f(x, y) = 1/3\}$ is the origin $(0, 0)$.
The level curve $\{(x, y) \in D : f(x, y) = 1/2\}$ is the circle $x^2 + y^2 = 5$ on the xy-plane centered at the origin and of radius $\sqrt{5}$.
The level curve $\{(x, y) \in D : f(x, y) = 1\}$ is the circle $x^2 + y^2 = 8$ on the xy-plane centered at the origin and of radius $\sqrt{8}$.

5. (10 pts) Let $z = f(x, y)$ be a differentiable function with equal partial derivatives everywhere, that is, $f_x(a, b) = f_y(a, b)$ for all (a, b) in $\mathbb{R}^2$. Consider the function

$$g(s, t) = f(s^2 - t^2, t^2 - s^2)$$

and find $\dfrac{\partial g}{\partial s}$ and $\dfrac{\partial g}{\partial t}$. What can you say about the graph of g? Explain.
Solution. Since $g = f(x, y)$, with $x = s^2 - t^2$ and $y = t^2 - s^2$, from the chain rule it follow that

$$\begin{aligned}\frac{\partial g}{\partial s} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} \\ &= \frac{\partial f}{\partial x} \cdot 2s + \frac{\partial f}{\partial y} \cdot (-2s) \\ &= 2s \cdot \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) \\ &= 0\end{aligned}$$

because $f_x = f_y$. Similarly,

$$\begin{aligned}\frac{\partial g}{\partial t} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} \\ &= \frac{\partial f}{\partial x} \cdot (-2t) + \frac{\partial f}{\partial y} \cdot 2t \\ &= 2t \cdot \left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\right) \\ &= 0\end{aligned}$$

Hence g must be a constant function. Indeed, consider any two points (a, b) and (c, d) in $\mathbb{R}^2$. We want to show that $g(a, b) = g(c, d)$. Intuitively, this is clear: the intersection of the graph of g with any vertical plane parallel to the s and t-axes is a line with zero slope. Hence we may "walk" on the graph of g from $(a, b, g(a, b))$ straight to $(a, d, g(a, d))$ on a line with zero slope, and then - if necessary, that is, if $c \neq a$ - we may walk on a line with zero slope straight to $(c, d, g(c, d))$. This feels like "walking" on a horizontal plane.

More rigorously, define the one-variable functions

$$h(s) = g(s,d) \quad \text{and} \quad k(t) = g(a,t),$$

and note that they are constant function of s and t, respectively, because

$$h'(s) = \frac{\partial g}{\partial s}(s,d) = 0 \quad \text{and} \quad k'(t) = \frac{\partial g}{\partial t}(a,t) = 0$$

Since the function k is constant, $g(a,b) = k(b) = k(d) = g(a,d)$, and since the function h is constant, $g(a,d) = h(a) = h(c) = g(c,d)$. Hence

$$g(a,b) = g(a,d) = g(c,d),$$

as desired. Therefore the graph of g must be a horizontal plane parallel to the st-plane.

6. (15 pts) Consider the following three lines

$$\begin{array}{lllll} L1: & x = 1+t & y = -2+3t & z = 4-t & -\infty < t < \infty \\ L2: & x = 2s & y = 3+s & z = -3+4s & -\infty < s < \infty \\ L3: & x = -1+2r & y = -3+r & z = 4r & -\infty < r < \infty \end{array}$$

(a) Show that L_1 and L_3 intersect, L_2 and L_3 are parallel, while L_1 and L_2 are skew.
Solution.

$\underline{L_1 \text{ and } L_3}$. We must solve the system

$$\begin{aligned} 1+t &= -1+2r \\ -2+3t &= -3+r \\ 4-t &= 4r \end{aligned}$$

Adding the first and the third equation gives us $5 = -1+6r$. Hence $r = 1$, and $t = 4 - 4r = 0$. The pair $(r,t) = (1,0)$ clearly satisfies the second equation, as well. Hence the two lines intersect at the point $P(1,-2,4)$.

$\underline{L_2 \text{ and } L_3}$. Both L_2 and L_3 are parallel to the vector $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$, and hence they are parallel.

$\underline{L_1 \text{ and } L_2}$. L_1 is parallel to the vector $\mathbf{v} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$. It is

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 13\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}.$$

Since $\mathbf{v} \times \mathbf{u} \neq \mathbf{0}$, the vectors $\mathbf{v}$ and $\mathbf{u}$ are not parallel, and so L_1 is not parallel to L_2. L_1 is does not intersect L_2, because the system

$$\begin{aligned} 1+t &= 2s \\ -2+3t &= 3+s \\ 4-t &= -3+4s \end{aligned}$$

has no solution. Indeed, adding the first and the third equation gives us $5 = -3 + 6s$. Hence $s = 8/6 = 4/3$, and $t = 2s - 1 = 5/3$. However, the pair $(s, t) = (4/3, 5/3)$ does satisfy the second equation

$$-2 + 3t = 3 \neq 17/3 = 3 + s.$$

Therefore L_1 and L_2 are skew.
Note: We could argue that the vectors $\mathbf{v}$ and $\mathbf{u}$ are not parallel by noticing that their components are not proportional. However, we still need the calculation of $\mathbf{v} \times \mathbf{u}$ in part (b).

(b) Find the plane determined by L_1 and L_3.
Solution. The plane determined by L_1 and L_3 passes through $P(1, -2, 4)$ and is normal to $\mathbf{v} \times \mathbf{u} = 13\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}$. Its equation is therefore

$$13(x - 1) - 6(y + 2) - 5(z - 4) = 0,$$

or equivalently

$$13x - 6y - 5z = 5.$$

(c) Find the distance between L_1 and L_2.
Solution. Since L_1 and L_2 are skew, the distance between them is equal to the distance between the two parallel planes M_1 and M_2 that contain L_1 and L_2, respectively, and are normal to $\mathbf{v} \times \mathbf{u}$. We found the equation of M_1 in part (b). Now it suffices to consider any point on L_2 and find its distance from M_1. Consider the point $S(0, 3, -3)$ on L_2 (i.e. set $s = 0$). The distance of S from M_1 equals

$$\frac{|13 \cdot 0 + (-6) \cdot 3 + (-5) \cdot (-3) - 5|}{\sqrt{13^2 + (-6)^2 + (-5)^2}} = \frac{8}{\sqrt{230}}.$$