Math 192 – Practice Prelim 2 – fall 2004

Gradients, Directional Derivatives, and Tangent Planes

1. Find the tangent plane to the surface $\frac{x^2}{2^2}+\frac{y^4}{2^4}+\frac{z^6}{2^6}=3$ at (2, 2, 2).

2. Find the directional derivative of $f(x,y)=2x^2+3xy-y^2$ at the point (2, 3) in the same direction as $\mathbf{v}=\mathbf{i}+3\mathbf{j}$.

3. (a) Find a point on the surface $4x^2+y^2+2z=25$ whose tangent plane is parallel to the plane $4x+3y+z=0$.

   (b) Write the equation for the tangent plane.

4. Consider $f(x,y,z)=xy+\ln z$ and the point $P(1, 0, 1)$.

   (a) Find the derivative of f at P in the direction $\mathbf{v}=\mathbf{i}-\mathbf{j}+2\mathbf{k}$. What is the direction in which f increases most rapidly at P? Find the tangent plane and the normal line at P to the surface $f(x,y,z)=0$.

   (b) Find parametric equations for the line tangent at P to the curve of intersection of the surfaces $f(x,y,z)=0$ and $2x^2-ze^y=1$.

5. At the point $P(1, 0)$ find a direction such that the directional derivative of $f(x,y)=x^2e^{-2y}$ is equal to 2.

6. The sphere $x^2+y^2+(z-3)^2=9$ rests between the two planes

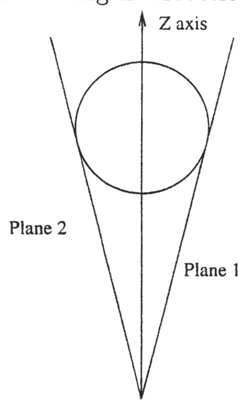
   $x-2y-2z=c$

   $-x+2y-2z=c$

   (a) At what points is the sphere tangent to the planes?

   (b) What is c?

Side view along intersection of planes.



Critical Points.

7. Consider the function $f(x,y)=2xy-x^2-2y^2+3x+4$.

   (a) Find all critical points. Find the local maxima and the local minima, and all points where they occur.

   (b) Find the tangent plane to the graph surface $z=f(x,y)$ at the point (1, 0, 6).

8. Consider $f(x, y) = k^2 e^x - xy \sin k + (6 - 5k)x$ where k is a real number.

   (a) Find all values of k for which (0, 0) is a critical point of the function f.

   (b) Can (0, 0) be a local extreme of f?

9. (a) Test $f(x, y) = x^3 + 3xy + y^3$ for local maxima, local minima, and saddle points.

   (b) Find the absolute maximum and minimum of $f(x, y) = x - 4xy + 2y + 1$ on the closed triangular plate in the first quadrant bounded by the lines $x = 0$, $y = 0$, and $y = 1 - x$.

10. Find all local maxima, local minima, and saddle points of $z = x^2 + y^2 - xy - x$.

11. Locate the critical points of the function $f(x, y) = x^4 + xy + y^4$ and classify them as local maximum points, local minimum points, or saddle points.

12. Consider the function $f(s,t) = 3st + s^3 - t^3$.

    (a) Find the local maxima, local minima, and saddle points for f.

    (b) Find the absolute maxima of the function $f(s,t)$ over the region bounded by the lines $s = t$, $s = 1$, and $t = 0$.

13. Find the greatest and smallest values of the function $f(x, y) = x^2 y$ in the region $x^2 + y^2 \le 1$.

14. The height above sea level of Gorgeous State Park is given by $H(x, y) = 4xy - x^4 - y^2 + 350$ for $-4 \le x \le 4$ and $-4 \le y \le 4$. Professor Pontificus goes for his daily walk in Gorgeous Park along a path given by

$$x(t) = t$$

$$y(t) = t^2$$

    Find the coordinates of all points on the path where the Professor will be neither ascending nor descending.

15. Consider the function $f(x, y) = x^2 + y^2 - 2x - 4y$ on the region R bounded by the lines $y = x$, $y = 3$, and $x = 0$. Find the absolute maximum and the absolute minimum of $f(x, y)$ on R.

16. Consider the function $f(x, y) = \int_x^y (3 - 2t - t^2)dt$ defined on the domain $x \le y$ (that is, the domain is the half-plane $x \le y$). Find the critical points of the function. What is the absolute maximum of f? At what points does the absolute maxima occur? Explain why the absolute maxima exist.

Double Integrals and Polar Coordinates

17. Consider the integral $f(x,y) = \int_{-1}^{1} \int_{x^2}^{\sqrt{2-x^2}} y \, dy \, dx$.

    (a) Describe or sketch the region of integration

    (b) Evaluate the integral

    (c) Reverse the order of integration.

18. Consider the integral $f(x,y) = \int_{0}^{1} \int_{y^2}^{y^{1/3}} xy^2 \, dx \, dy$.

    (a) Evaluate the integral.

    (b) Sketch the region of integration.

    (c) Reverse the order of integration.

19. Reverse the order of integration in the following double integrals. DO NOT evaluate the integrals.

    (a) $\int_{0}^{\pi/2} \int_{0}^{\sin x} (x \sin y) \, dy \, dx$ (b) $\int_{1}^{2} \int_{\ln x}^{e^x} (x^2 + y) \, dy \, dx$

20. Consider the integral

$$\int_{0}^{2} \int_{0}^{4-x^2} \frac{xe^{2y}}{4-y} \, dy \, dx$$

    (a) Sketch the region of integration.

    (b) Reverse the order of integration.

    (c) Evaluate the integral.

21. Consider the integral

$$\int_{0}^{1} \int_{0}^{\sqrt{1-x^2}} e^{-(x^2+y^2)} \, dy \, dx$$

    (a) Change the Cartesian integral into an equivalent polar integral.

    (b) Evaluate the integral.

22. For each of the following integrals: *i*) describe (in words or with a sketch) the region of integration and *ii*) evaluate the integral.

    (a) $\int_{0}^{3} \int_{0}^{y} (x^2 + y^3) \, dx \, dy$ (b) $\int_{0}^{\sqrt{\pi}} \int_{0}^{y^2} y\pi \cos x \, dx \, dy$ (c) $\int_{0}^{4} \int_{\sqrt{y}}^{2} ye^{-x^5} \, dx \, dy$

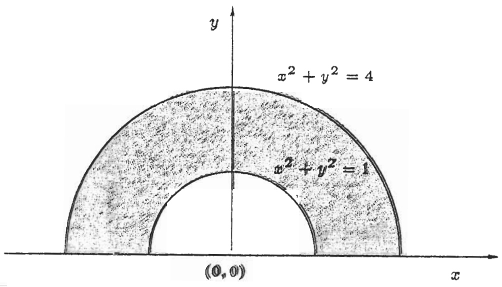
23.(a) Consider the integral $\int_0^1 \int_{x^2}^{\sqrt{x}} f(x,y)\,dy\,dx$. Sketch the region of integration. Reverse the order of integration. Evaluate the integral for $f(x,y) = \sqrt{y} + y^2$.

(b) Consider the integral $\int_0^1 \int_{x^2+p}^{\sqrt{x}+p} f(x,y)\,dy\,dx$ where p is a positive number. Sketch the region of integration. Reverse the order of integration. Use your results in (a) to evaluate the integral when $f(x,y) = \sqrt{y-p} + (y-p)^2$. Explain.

24. Find the area of one leaf of the rose $r = \sin(3\theta)$.

25. Sketch the curve $r = \sin(2\theta)$. Find the intersection points of the curve and the circle $r = \cos\theta$.

26. Consider the half-annulus given by $y \geq 0$ and $1 \leq (x^2 + y^2) \leq 4$ as seen in the figure. The density is uniform. **Calculate the *y* coordinate of the center of mass.**



27. Find the centroid (i.e., **the center of mass with uniform density**) of the region bounded by $y = 1 - x^2$ and the x-axis.

Triple Integrals

28. Calculate the volume of the cylinder $x^2 + y^2 \leq 1$ between the planes $z = x$ and $z = 2 + 2x$.

29. A plane cuts a sphere of radius R. The plane is at a distance h from the center of the sphere. (We assume that $R > h > 0$.) Set up and evaluate a triple integral computing the volume of the region that contains the center of the sphere, and is bounded by the plane and by the sphere.

30. Set up a triple integral (or a sum of triple integrals) to compute the volume of the region bounded below by the plane $z = 0$, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$. Use cylindrical coordinates and the order of integration $dr\,dz\,d\theta$. Sketch the region. Do not evaluate the integral.