Solutions to the Practice Problems for the Final, Fall 2006

1. Find the plane that contains the lines $L_1: \ x = 1 + t, \ y = 1 - t, \ z = -t$ and $L_2: \ x = 1 + 3t, \ y = -3t, \ z = -3 - 3t$.

   SOLUTION: The lines are parallel. We choose two points on L_1, say $(1, 1, 0)$ with $t = 0$ and $(2, 0, -1)$ with $t = 1$. We also choose a point on L_2, say $(1, 0, -3)$ with $t = 0$. The plane goes through the three points, so it is $2x + 3y - z = 5$.

2. Find and classify the critical points of the function $f(x, y) = 4x^2e^y - 2x^4 - e^{4y}$.

   SOLUTION: An easy computation shows that the first order partial derivatives of f exist at every point. So the only critical points for f are the ones in which both f_x and f_y vanish:

$$\left\{\begin{array}{llllll} f_x = 8xe^y - 8x^3 = 0 & \Leftrightarrow & 8x(e^y - x^2) = 0 & \Leftrightarrow & x = 0 \text{ or } e^y = x^2 & (1) \\ f_y = 4x^2e^y - 4e^{4y} = 0 & \Leftrightarrow & 4e^y(x^2 - e^{3y}) = 0 & \Leftrightarrow & e^{3y} = x^2 & (2) \end{array}\right.$$

   We notice that $x = 0$ is not a solution of equation (2), and we solve $\left\{\begin{array}{l} e^y = x^2 \\ e^{3y} = x^2. \end{array}\right.$ This system gives:

$$e^y = e^{3y} \Leftrightarrow e^y(1 - e^{2y}) = 0 \Leftrightarrow e^{2y} = 1 \Leftrightarrow y = 0$$

   and $x^2 = e^0 = 1$, hence $x = \pm 1$.
   There are two critical points: $P = (1, 0)$ and $Q = (-1, 0)$. To classify them as max/min/saddle, we compute the second order partial derivatives:

$$\left\{\begin{array}{l} f_{xx} = 8e^y - 24x^2 \\ f_{yy} = 4x^2e^y - 16e^{4y} \\ f_{xy} = 8xe^y. \end{array}\right.$$

   At $P = (1, 0)$, we have $\det\begin{pmatrix} -16 & 8 \\ 8 & -12 \end{pmatrix} = 128 > 0$, and $f_{xx} = -16 < 0$, so $P = (1, 0)$ is a local maximum.
   At $Q = (-1, 0)$, we have $\det\begin{pmatrix} -16 & -8 \\ -8 & -12 \end{pmatrix} = 128 > 0$, and $f_{xx} = -16 < 0$, so $Q = (-1, 0)$ is also a local maximum.

3. Find the absolute maximum of the function $f(x,y,z) = x^2yz$ on the triangle cut by the plane $x+y+z=12$ and the first octant.

   SOLUTION: We apply the method of Lagrange multipliers, with $f(x,y,z) = x^2yz$ and $g(x,y,z) = x+y+z-12$. Since

   $$\nabla(f) = \langle 2xyz, x^2z, x^2y\rangle \quad \text{and} \quad \nabla(g) = \langle 1,1,1\rangle$$

   we need to solve the system

   $$\begin{cases} 2xyz = \lambda & (1) \\ x^2z = \lambda & (2) \\ x^2y = \lambda & (3) \\ x+y+z = 12 & (4). \end{cases}$$

   From equations (2) and (3) we get $x^2z = x^2y$, so $x = 0$ or $z = y$.

   First, note that the value of f is 0 at any point on the boundary of the triangle. Now assume that $x \neq 0$, $y \neq 0$, and $z \neq 0$. Hence $z = y$, so $\lambda = x^2y$ and equation (1) gives

   $$2xyz = x^2y \Leftrightarrow xy(2z - x) = 0 \Rightarrow x = 0 \text{ or } y = 0 \text{ or } x = 2z.$$

   So, $x = 2z$ (and again $z = y$). Plugging everything in equation (4), we get:

   $$x+y+z = 12 \Leftrightarrow 2z+z+z = 12 \Leftrightarrow z = 3$$

   so $x = 6$ and $y = 3$. We obtain the solution $Q = (6,3,3)$, which gives $f(Q) = 36\cdot3\cdot3 = 324$.

   We conclude that the absolute maximum value of f on the triangle is 324, and it occurs at the point $Q = (6,3,3)$.

4. A thin wire lies along the curve $\mathbf{r}(t)$, $0 \leq t \leq 1$. The wire density δ (its mass per unit length) increases linearly along the wire and is $\delta(\mathbf{r}) = s+1$, where s is the arc length parameter, and $s = 0$ corresponds to the $t = 0$ endpoint of the wire. Determine the wire mass if $|\mathbf{v}(t)| = t^2+1$.

   SOLUTION:

   $$s(t) = \int_0^t |\mathbf{v}(t)|dt = \int_0^t t^2 + 1\ dt = t^3/3 + t$$

   $$M = \int_0^1 \delta(\mathbf{r})|\mathbf{v}(t)|dt = \int_0^1 (t^3/3 + t + 1)(t^2+1)\ dt = 20/9\,.$$

5. Find the flux of the field $\mathbf{F} = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ outward through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0$, $x = 1$, and $z = 0$.

   SOLUTION:

   $$g = z + y^2 - 4,\ \nabla g = 2y\mathbf{j} + \mathbf{k},\ |\nabla g| = \sqrt{4y^2+1},\ \mathbf{n} = \frac{2y\mathbf{j}+\mathbf{k}}{\sqrt{4y^2+1}},\ \ \mathbf{F}\cdot\mathbf{n} = \frac{2xy - 3z}{\sqrt{4y^2+1}}\ .$$

   $$\mathbf{p} = \mathbf{k},\ |\nabla g \cdot \mathbf{p}| = 1,\ d\sigma = \sqrt{4y^2+1}\, dA\,.$$

   $$\mathrm{F}lux = \iint\limits_R \frac{2xy - 3z}{\sqrt{4y^2+1}}\sqrt{4y^2+1}\, dA = \iint\limits_R 2xy - 3z\ dA = \iint\limits_R 2xy - 3(4-y^2)\ dA$$

   $$= \iint\limits_R 2xy - 12 + 3y^2\ dA = \int_0^1\int_{-2}^2 2xy - 12 + 3y^2\ dydx = -32\,.$$

6. Consider the gravitational field:

   $$\mathbf{F} = -GMm\frac{(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{(x^2+y^2+z^2)^{3/2}}\ ,$$

   where G, M, m are constants.

   (a) Show that the field is conservative. (Hint: Set $r = \sqrt{x^2+y^2+z^2}$, and use that $\frac{\partial r}{\partial x} = \frac{x}{r}$.)

   (b) Find a potential function $f(x, y, z)$ for F. (Hint: Try f of the form C/r where C is a constant.)

   (c) Let P and Q be points at distances s and t from the origin. Find the work done by the gravitational field $\mathbf{F}$ in moving a particle from P to Q, i.e. evaluate the integral $\displaystyle\int_P^Q \mathbf{F}\cdot d\mathbf{r}$.

   SOLUTION: Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ and set $a = GMm$.

   (a) $F_1 = -\frac{ax}{r^3}$ and $F_3 = -\frac{az}{r^3}$. Therefore:

   $$\frac{\partial F_1}{\partial z} = \frac{3ax}{r^4}\frac{\partial r}{\partial z} = \frac{3axz}{r^5}\ .$$

   $$\frac{\partial F_3}{\partial x} = \frac{3az}{r^4}\frac{\partial r}{\partial x} = \frac{3axz}{r^5}\ .$$

   The other two parts of the test can be verified in analogous way.

   (b) Try $f = C/r$. Now: $\dfrac{\partial f}{\partial x} = -\dfrac{Cx}{r^3},\ \dfrac{\partial f}{\partial y} = -\dfrac{Cy}{r^3},\ \dfrac{\partial f}{\partial z} = -\dfrac{Cz}{r^3}$. Therefore, $C = a$ and $f = GMm/r$.

   (c)

   $$\mathrm{W}ork = \int_P^Q \mathbf{F}\cdot d\mathbf{r} = f(Q) - f(P) = GMm\left(\frac{1}{t} - \frac{1}{s}\right)\ .$$

7. Let **F** be the vector field

$$\mathbf{F}(x,y,z)=(x^2+y+zy^3)\mathbf{i}+ax(y+1)\mathbf{j}+(x^3+y^2+xyz^3)\mathbf{k}$$

where a is a real number. Find the parameters a such that the flux of *curl* **F** across the unit upper hemisphere is zero.

SOLUTION: The flux is the circulation of **F** along the unit circle in the xy-plane. This circulation is equal to the flux of $Curl(\mathbf{F})$ across the unit disk, which turns out to be

$$\int\int_{Disk} a(y+1)-1\ dxdy=\pi(a-1)\,,$$

and the answer is $a=1$. It is also possible to compute directly the circulation (this is not difficult).

8. Let

$$f(x,y,z)=(x^2+y^2+z^2)^a$$

   (a) Compute div(grad(f)).

   (b) For which value(s) of a is div(grad(f)) $\equiv 0$?

   SOLUTION:

$$\begin{aligned}\text{grad}(f)&=2a(x^2+y^2+z^2)^{a-1}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\\ \text{div}(\text{grad}(f))&=2a*(2(a-1)+3)(x^2+y^2+z^2)^{a-1}\end{aligned}$$

   This vanishes when $a=0$ and when $a=-1/2$.

9. (a) Graph the part of the spiral $r=\theta$ for $0\le\theta\le 8\pi$.

   (b) Calculate the area of the shaded region, shown in the picture.

SOLUTION: (b) Area $=\displaystyle\int_0^{2\pi}\int_\theta^{\theta+2\pi} r\,dr\,d\theta=2\pi\int_0^{2\pi}(\theta+\pi)d\theta=8\pi^3\,.$

Math 192 Final Exam Solutions. Dec 8, 2005

1. $f(x,y)=\sqrt{1-\frac{x^2}{2}-\frac{y^2}{16}} \Rightarrow f_x=\frac{-x}{2}\left(1-\frac{x^2}{2}-\frac{y^2}{16}\right)^{-\frac{1}{2}}, f_y=\frac{-y}{16}\left(1-\frac{x^2}{2}-\frac{y^2}{16}\right)^{-\frac{1}{2}}$,
$f_x(1,2)=-1, f_y(1,2)=-\frac{1}{4}, f(1,2)=\frac{1}{2}$

a) Equation of tangent plane: $-1(x-1)-\frac{1}{4}(y-2)-(z-\frac{1}{2})=0$.

b) Linearization:
$L(x,y)=f(1,2)+f_x(1,2)(x-1)+f_y(1,2)(y-2)=\frac{1}{2}-(x-1)-\frac{1}{4}(y-2),\ L(0.8,2.4)=0.6$

2. a) Since the components of $\vec{F}$ and their partial derivatives are continuous everywhere, we may apply Green's Theorem with $M=-y$ and $N=x$ to get

$$\int_C(\vec{F}\cdot\vec{T})ds=\int\int_D[1-(-1)]dA=2(\text{Area of } D).$$

b) $\vec{F}\cdot\frac{d\vec{r}}{dt}=(b\sin t)(a\sin t)+(a\cos t)(b\cos t)=ab$. Therefore, $\int_C(\vec{F}\cdot\vec{T})ds=\int_0^{2\pi}(\vec{F}\cdot\frac{d\vec{r}}{dt})dt=\frac{1}{2}\int_0^{2\pi}(ab)dt=2\pi ab$. Thus, area of ellipse $=\frac{1}{2}(2\pi ab)=\pi ab$.

3. The circles intersect when $2\sin\theta=1\Rightarrow\sin\theta=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{6},\frac{5\pi}{6}$. Therefore, area $=\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_1^{2\sin\theta}\ rdrd\theta=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}(4\sin^2\theta-1)d\theta=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}(1-2\cos 2\theta)d\theta=\frac{1}{2}[\theta-\sin 2\theta]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}=\frac{\pi}{3}+\frac{\sqrt{3}}{2}$.

4. Since $z=\rho\cos\phi$, the plane and sphere intersect when $1=2\cos\phi\Rightarrow\phi=\frac{\pi}{3}$. Also, $\rho=\sec\phi$ everywhere on the plane. Therefore, Volume $=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_{\sec\phi}^2\rho^2\sin\phi\ d\rho d\phi d\theta=\frac{1}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{3}}[8-\sec^3\phi]\sin\phi\ d\phi d\theta=\frac{1}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{3}}[8\sin\phi-\tan\phi\sec^2\phi]\ d\phi d\theta=\frac{5\pi}{3}$.

5. $f(x,y)=x^2-xy$. There are no critical points in region R. The boundary is $x^2+y^2-xy=1$. Let $g(x,y)=x^2+y^2-xy$. Then $\nabla f=\lambda\nabla g\Rightarrow 2x-y=\lambda(2x-y), -x=\lambda(2y-x)\Rightarrow\lambda=1$ or $y=2x$.
$\lambda=1\Rightarrow -x=2y-x\Rightarrow y=0, x=1,-1$. But $(1,0),(-1,0)$ are not in R.
$y=2x\Rightarrow x^2+(2x)^2-x(2x)=1\Rightarrow x=\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}}$. The point $\left(\frac{1}{\sqrt{3}},\frac{2}{\sqrt{3}}\right)$ is in R. On $y=1, f(x,1)=x^2-x$ on $[0,1]$. This has critical point $(\frac{1}{2},1)$ and endpoints of this interval in R are $(0,1)$ and $(1,1)$. So f must be evaluated at: $(0,1),(1,1),(\frac{1}{2},1),(\frac{1}{\sqrt{3}},\frac{2}{\sqrt{3}})$. The maximum value of f on R is $f(1,1)=f(0,1)=0$ and the minimum value is $f(\frac{1}{\sqrt{3}},\frac{2}{\sqrt{3}})=\frac{-1}{3}$.

6. Average distance of a point on S from the plane $=\left(\frac{1}{area}\right)\int\int_S z\ d\sigma$, where $d\sigma=\frac{|\nabla f|}{|\nabla f\cdot\vec{k}|}\ dA$.
$f(x,y,z)=x^2+y^2+z^2\Rightarrow\nabla f=2x\vec{i}+2y\vec{j}+2z\vec{k}\Rightarrow|\nabla f\cdot\vec{k}|=2z\Rightarrow\frac{|\nabla f}{|\nabla f\cdot\vec{k}|}=\frac{1}{z}$.
Therefore, $\int\int_S z\ d\sigma=\int\int_R z\left(\frac{1}{z}\right)\ dA$ where R is the region inside the circle $r=\cos\theta$ of radius 1/2. So, $\int\int_S z\ d\sigma=\frac{\pi}{4}$. Now, area of surface S $=\int\int_S d\sigma=\int\int_R\frac{1}{z}\ dA=\int_0^{\pi}\int_0^{\cos\theta}(1-r^2)^{-1/2}\ r\ drd\theta$
$=-\int_0^{\pi}(1-r^2)^{1/2}]_0^{\cos\theta}\ d\theta=-\int_0^{\pi}(\sin\theta-1)\ d\theta=\pi-2$. Thus, average distance $=\frac{\pi/4}{\pi-2}=\frac{\pi}{4\pi-8}$.

7. a) Two pairs of partials are already equal. Solve $\frac{\partial P}{\partial y}=2a^2z=\frac{\partial N}{\partial z}=2z+1-a^2\Rightarrow a=1$ or -1.
b) $\frac{\partial f}{\partial x}=M=e^x\cos y\Rightarrow f=e^x\cos y+g(y,z)$. $\frac{\partial f}{\partial y}=-e^x\sin y+\frac{\partial g}{\partial y}=N=-e^x\sin y+z^2\Rightarrow\frac{\partial g}{\partial y}=z^2\Rightarrow g(y,z)=yz^2+h(z)$. $\frac{\partial f}{\partial z}=2yz+h'(z)=P=2yz.\Rightarrow h(z)=C$. Therefore, $f(x,y,z)=e^x\cos y+yz^2+C$.
c) Work along curve C $=\int_C\vec{F}\cdot d\vec{r}=\int_A^B df=f(1,\pi/2,2)-f(0,0,7)=2\pi-1$.

8. Let S_1 be the surface of the cylinder for $y\geq 0$. $f(x,y)=x^2+y^2\Rightarrow\nabla f=2x\vec{i}+2y\vec{j}\Rightarrow|\nabla f|=2\sqrt{x^2+y^2}=4$. So, $d\sigma=\frac{|\nabla f|}{|\nabla f\cdot\vec{j}|}\ dA=\frac{4}{2y}\ dA$. Therefore, $\vec{F}\cdot\vec{n}\ d\sigma=\vec{F}\cdot\left(\frac{2x\vec{i}+2y\vec{j}}{4}\right)\left(\frac{4}{2y}\right)\ dA=\frac{2x^2+2y^2}{2y}\ dA$

$= \frac{4}{y}\, dA$. Thus, flux $= \int\int_{S_1} \vec{F}\cdot\vec{n}\, d\sigma = \int\int_R \frac{4}{y}\, dA = \int_0^3 \int_{-2}^2 4(4-x^2)^{-1/2}\, dxdz = 4\int_0^3 \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^2 dz = 12\pi$. Similarly, the surface S_2 of the cylinder for $y \leq 0$ also has flux 12π. Therefore, total flux across the cylinder $= 24\pi$.

9. For the line integral around C, we use $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j}$, $\frac{\vec{dr}}{dt} = -\sin t\vec{i} + \cos t\vec{j}$. On C we have $z = 0$ and so $\vec{F} = (x^2+y^2)\vec{k}$. Therefore, $\vec{F}\cdot\frac{\vec{dr}}{dt} = 0 \Rightarrow \int_C \vec{F}\cdot\vec{dr} = 0$.
For the surface integral, the curl $\nabla\times\vec{F} = y\vec{i} - x\vec{j}$. and $\nabla f = 2x\vec{i} + 2y\vec{j} + 5\vec{k}$. So $(\nabla\times\vec{F})\cdot\vec{n}\, d\sigma = (y\vec{i} - x\vec{j})\cdot\left(\frac{2x\vec{i}+2y\vec{j}+5\vec{k}}{|\nabla f|}\right)\left(\frac{|\nabla f|}{|\nabla f\cdot\vec{k}|}\right) dA = \frac{(2xy-2xy)}{5}\, dA = 0$. Therefore, the flux of the curl of $\vec{F} = 0$.

10. By the Divergence Theorem, $\int\int_S \vec{F}\cdot\vec{n}\, d\sigma = \int\int\int_D div\vec{F}\, dV = \int_0^{2\pi}\int_0^1\int_1^{e^{1-r^2}} (1)\, dz\, rdrd\theta = \int_0^{2\pi}\int_0^1 \left(e^{1-r^2} - 1\right) rdrd\theta = \frac{-1}{2}\int_0^{2\pi}\left[e^{1-r^2} + r^2\right]_0^1 d\theta = \pi(e-2)$.