Math 192 – Solutions to Practice Final Exam – Fall 2004

1. (a) $\frac{5}{77}$.
   (b) The region is bounded above by $x = y^2$ and below by $x = y^{1/3}$.
   (c) $\int_0^1 \int_{x^3}^{x^{1/2}} xy^2 dydx$.

2. $\int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^1 rdrdzd\theta + \int_0^{2\pi} \int_{\sqrt{3}}^2 \int_0^{\sqrt{4-z^2}} rdrdzd\theta$.

3. $\int_0^{2\pi} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^2 \rho^2 \sin\phi \cos^2\phi d\rho d\phi d\theta = \frac{2\pi}{9}$.

4. (a) $(9, -16, 4)$.
   (b) $x - y - 2z = 17$.

5. $\mathbf{r}(t) = (-\frac{t^2}{2} + 10)\mathbf{i} + (-\frac{t^2}{2} + 2)\mathbf{j} + (-t^3 + 5)\mathbf{k}$, which is $(8, 0, -3)$ at $t = 2$.

6. (a) A unique local maximum $f(3, \frac{3}{2}) = \frac{17}{2}$.
   (b) $x + 2y - z = 7$.

7. (a) Linearization: $6 + 4(x-1) + 5(y-2)$.
   (b) $M = 4.3$, so $|E| \leq 0.086$.

8. Use cylindrical coordinates: the center of the sphere is at the origin, the plane has equation $z = h$. The volume is given by the integral $\int_{-R}^{h} \int_0^{2\pi} \int_0^{\sqrt{R^2-z^2}} rdrd\theta dz = \pi(\frac{2}{3}R^3 + R^2h - \frac{h^3}{3})$.

9. (20 points) Find all local minima, local maxima, and saddle points of $z = x^2 + y^2 - xy - x$.

SOLUTION: We have:

$$f(x,y) = x^2 + y^2 - xy - x \qquad f_{xx}(x,y) = 2$$
$$f_x(x,y) = 2x - y - 1 \qquad f_{yy}(x,y) = 2$$
$$f_y(x,y) = 2y - x \qquad f_{xy}(x,y) = -1$$

Our function is a polynomial so the gradient exists everywhere, so our only critical points occur when the gradient is **0**. This happens when:

$$\begin{aligned} y &= 2x - 1 \\ y &= \frac{x}{2} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x &= \frac{2}{3} \\ y &= \frac{1}{3} \end{aligned}$$

So our only critical point is $\left(\frac{2}{3}, \frac{1}{3}\right)$. Now we must check if it a local min, local max, or a saddle point. We have

$$f_{xx}f_{yy} - f_{xy}^2 = 4 - 1 = 3 > 0$$

so our critical point is not a saddle point. Since $f_{xx}(2/3, 1/3) = 2$ we have that our critical point is a local minimum by the 2nd derivative test.

10. (20 points) Find the area of one leaf of the rose $r = \sin(3\theta)$.

SOLUTION: The r limits of integration are 0 to $\sin(3\theta)$, and the θ limits of integration are 0 to $\pi/3$. So we have the area is given by the integral:

$$\int_0^{\pi/3} \int_0^{\sin(3\theta)} r\, dr\, d\theta = \int_0^{\pi/3} \frac{\sin^2(3\theta)}{2} d\theta = \frac{1}{2} \int_0^{\pi/3} \frac{1 - \cos(6\theta)}{2} d\theta = \frac{1}{4}(\pi/3) = \pi/12$$

11. Consider the integral $\int_{-1}^{1}\int_{x^2}^{\sqrt{2-x^2}} y\,dy\,dx$.

(a) (5 points) Describe or sketch the region of integration.
(b) (5 points) Evaluate the integral.
(c) (10 points) Reverse the order of integration.

Solution: (a) The region D of integration is bounded below by the parabola $y = x^2$ and above by the circle $x^2 + y^2 = 2$.

(b) We evaluate the integral:

$$\int_{-1}^{1}\int_{x^2}^{\sqrt{2-x^2}} y\,dy\,dx = \int_{-1}^{1} \frac{1}{2}[(2-x^2) - x^4]dx = \frac{1}{2}[2x - x^3/3 - x^5/5]\Big|_{-1}^{1} = 2 - 1/3 - 1/5 = \frac{22}{15}.$$

(c) Reversing the order of integration, we get $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} y\,dx\,dy + \int_1^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} y\,dx\,dy.$

12. Set up triple integrals to compute the volumes of the given regions. Do not evaluate the integrals.

(a) (8 points) Let D be the region bounded above by the paraboloid $z = 8 - x^2 - y^2$ and below by the paraboloid $z = x^2 + y^2$. Use cartesian coordinates and the order of integration $dydzdx$.
(b) (9 points) Let D be the region bounded below by the xy-plane, on the sides by the sphere $\rho = \sqrt{2}$, and above by the cone $z = \sqrt{x^2+y^2}$. Use cylindrical coordinates and the order of integration $dzdrd\theta$.
(c) (8 points) Let D be the region bounded above by the sphere of radius 2 centered at $(0, 0, 0)$ and below by the cone $z = \sqrt{x^2+y^2}$. Use spherical coordinates and the order of integration $d\rho d\phi d\theta$.

Solution: (a) $\int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dydzdx + \int_{-2}^{2}\int_{4}^{8-x^2}\int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dydzdx$

(b) $\int_0^{2\pi}\int_0^1\int_0^r r\,dzdrd\theta + \int_0^{2\pi}\int_1^{\sqrt{2}}\int_0^{\sqrt{2-r^2}} r\,dzdrd\theta$

(c) $\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^2 \rho^2 \sin\phi\, d\rho d\phi d\theta$

13. (20 points) Calculate the volume of the cylinder $x^2 + y^2 \le 1$ between the planes $z = x$ and $z = 2 + 2x$.

SOLUTION: We have $x \le x + 1$ so $x \le 2(x + 1)$ and thus we have that our region is bounded on the bottom by $z = x$ and on the top by $z = 2 + 2x$. So our z limits of integration are from x to $2 + 2x$.

Change this to cylindrical coordinates and we get the z limits are from $r\cos\theta$ to $2 + 2r\cos\theta$.

Our r limits of integration are from 0 to 1, and our θ limits of integration are from 0 to 2π. So we get the volume is given by the integral:

$$\int_0^{2\pi}\int_0^1\int_{r\cos\theta}^{2+2r\cos\theta} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2r + r^2\cos\theta\,dr\,d\theta = \int_0^{2\pi} 1 + \frac{1}{3}\cos\theta\,d\theta = 2\pi$$

14. (20 points) Find the mass of a spherical ball of radius R with density

$$\delta(x, y, z) = k(\text{distance from center of ball to } (x, y, z)).$$

SOLUTION: In spherical coordinates we get the density function is $\delta(\rho, \phi, \theta) = k\rho$, and so our integral is:

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 k\rho^3\sin(\phi)\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/2} \frac{k}{4}\sin(\phi)\,d\phi\,d\theta = \int_0^{2\pi} \frac{k}{4}\,d\theta = \frac{k\pi}{2}$$

Some people attempted the problem before the correction was handed out. The density function was described as $\delta(x, y) = k(\text{distance from center of ball to } (x, y))$ which they interpreted as $\delta(x, y, z) = k(\text{distance from center of ball to } (x, y, 0))$. This makes the problem harder, since the density function, in cylindrical coordinates, is now $\delta(r, \theta, z) = kr$. The solution is below:

$$\int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} kr^2\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 kr^2\sqrt{1-r^2}\,dr\,d\theta = 2k\pi\int_0^2 r^2\sqrt{1-r^2}\,dr$$

Let $r=\sin\psi$, then $dr=\cos\psi\,d\psi$, and the limits become 0 and $\pi/2$ and so our integral becomes:

$$2k\pi\int_0^{\pi/2}\sin^2\psi\cos^2\psi\,d\psi = k\pi\int_0^{\pi/2}\sin^2(2\psi)\,d\psi = k\pi\int_0^{\pi/2}\frac{1-\cos(4\psi)}{2}\,d\psi = \frac{k\pi^2}{4}$$

15. (20 points) The following two non-intersecting lines are given: $L1: x=2+3t,\ y=1-4t,\ z=3-2t$ and $L2: x=3+3s,\ y=-2+s,\ z=5-s$. Find two parallel planes, such that $L1$ lies in one of the planes and $L2$ lies in the other plane.

Solution: The vector $\mathbf{v}_1=3\mathbf{i}-4\mathbf{j}-2\mathbf{k}$ is parallel to $L1$, and $\mathbf{v}_2=3\mathbf{i}+\mathbf{j}-\mathbf{k}$ is parallel to $L2$. The vector $\mathbf{v}_1\times\mathbf{v}_2=6\mathbf{i}-3\mathbf{j}+15\mathbf{k}$ is normal to both planes. The planes are $2x-y+5z=18$ and $2x-y+5z=33$.

16. Consider the function $f(s,t)=\displaystyle\int_s^t(6-x-x^2)dx$ defined in the domain $s\le t$ (that is, the domain is the unbounded half-plane $s\le t$).

(a) (10 points) Find the critical points of the function. Find the local maxima and local minima, and the points where they occur.

(b) (15 points) Does the absolute maximum exist? Explain.

Solution: (a) $(-3,2)$ is the only interior critical point. The second derivative test shows that a local maximum occurs at $(-3,2)$ and is equal to $\frac{125}{6}$.

(b) The function measures the area between the parabola $y=6-x-x^2$ and the x-axis. The parabola intersects the x-axis at the points -3 and 2. Hence, the absolute maximum is equal to $\int_{-3}^2(6-x-x^2)dx=\frac{125}{6}$.

17. (a) (12 points) Calculate the directional derivative of the function $f(x,y)=e^{x^2+y^2}$ at the point $(2,1)$ in the direction of $\mathbf{i}-\mathbf{j}$. Find an equation of the tangent plane to the surface $z=e^{x^2+y^2}$ at the point $(2,1,e^5)$.

(b) (13 points) If $\mathbf{r}=x\,\mathbf{i}+y\,\mathbf{j}$ and $f(x,y)=\ln\sqrt{x^2+y^2}$, write an expression for ∇f in terms of $\mathbf{r}$.

Solution: (a) $\nabla f=2xe^{x^2+y^2}\mathbf{i}+2ye^{x^2+y^2}\mathbf{j}$, so $\nabla f\big|_{(2,1)}=4e^5\mathbf{i}+2e^5\mathbf{j}$. The directional derivative is $D_{\mathbf{i}-\mathbf{j}}f\big|_{(2,1)}=\sqrt{2}e^5$. The tangent plane is $4e^5x+2e^5-z=9e^5$.

(b) $\nabla f=\dfrac{\mathbf{r}}{|\mathbf{r}|^2}$.

18. (25 points) Wire of total length 1000 cm is formed into a flexible coil in the shape of the circular helix $x=3\cos t,\ y=3\sin t,\ z=bt$, where b is a constant. There are 10 turns to each centimeter of height and the radius of the helix is 3 cm. How tall is the coil?

Solution: The arc length is $s=\sqrt{9+b^2}t$ since $ds=\sqrt{9+b^2}dt$ and we assume that $s=0$ at $t=0$. As we have that there are 10 turns to each centimeter of height, it follows that $b=\dfrac{1}{20\pi}$. Hence, $s=\sqrt{9+(20\pi)^{-2}}t$. The total length of the wire is $s=1000$, which occurs at $t=\frac{1000}{\sqrt{9+(20\pi)^{-2}}}$. Therefore, the coil is $z=bt=\frac{50}{\pi\sqrt{9+(20\pi)^{-2}}}$ tall.

19. Let $f(x,y)=x^2\sin y$.

a) (12 points) Find the linearization for $f(x,y)$ about the point $(1,\pi/2)$ and use your answer to approximate $f(.9,\pi/2+.2)$.

b) (12 points) Find an upper bound for the error in your estimate from part a.

SOLUTION: a) We have

$$\begin{aligned} f_x(x,y)&=2x\sin y\\ f_y(x,y)&=x^2\cos y\end{aligned} \quad\Longrightarrow\quad \begin{aligned} f_x\left(1,\frac{\pi}{2}\right)&=2\\ f_y\left(1,\frac{\pi}{2}\right)&=0\end{aligned}$$

So our linearization is

$$L(x,y)=f\left(1,\frac{\pi}{2}\right)+2(x-1)+0\left(y-\frac{\pi}{2}\right)=2x-1$$

Using this to approximate, we get $f\left(.9, \frac{\pi}{2} + .2\right) \approx .8$

b) We have

$$\begin{aligned} f(x,y) &= x^2 \sin y \\ f_x(x,y) &= 2x \sin y \\ f_y(x,y) &= x^2 \cos y \\ f_{xx}(x,y) &= 2\sin y \\ f_{yy}(x,y) &= -x^2 \sin y \\ f_{xy}(x,y) &= 2x \cos y \end{aligned} \quad \Longrightarrow \quad \begin{aligned} |f(.9,\pi/2+.2)| &\le (.81)(1) = .81 \\ |f_x(.9,\pi/2+.2)| &\le (1.8)(1) = 1.8 \\ |f_y(.9,\pi/2+.2)| &\le (.81)(1) = .81 \\ |f_{xx}(.9,\pi/2+.2)| &\le (2)(1) = 2 \\ |f_{yy}(.9,\pi/2+.2)| &\le (.81)(1) = .81 \\ |f_{xy}(.9,\pi/2+.2)| &\le (1.8)(1) = 1.8 \end{aligned}$$

So we see the bound on our second derivatives is $M = 2$. So we get an error bound of

$$|E| \le \frac{1}{2}(2)(.1 + .2)^2 = .09$$

20. A particle's velocity is given by $\mathbf{v}(t) = t\mathbf{i} + t^2\mathbf{j}$.
a) (12 points) If the particle's position at time $t = 0$ is $(2, 1)$, what is the particle's position at time $t = 1$?
b) (12 points) How far does the particle travel from time $t = 0$ to $t = 1$?

SOLUTION: **a)** We integrate

$$\int \mathbf{v}(t)\,dt = \int t\mathbf{i} + t^2\mathbf{j}\,dt = \frac{t^2}{2}\mathbf{i} + \frac{t^3}{3}\mathbf{j} + \mathbf{C} = \mathbf{r}(t)$$

Our initial condition of $\mathbf{r}(0) = 2\mathbf{i} + \mathbf{j}$ gives us that $\mathbf{C} = 2\mathbf{i} + \mathbf{j}$ and so our vector function which gives the particle's position is:

$$\mathbf{r}(t) = \left(\frac{t^2}{2} + 2\right)\mathbf{i} + \left(\frac{t^3}{3} + 1\right)\mathbf{j}$$

So $\mathbf{r}(1) = \frac{5}{2}\mathbf{i} + \frac{4}{3}\mathbf{j}$.

b) We have $|\mathbf{v}(t)| = \sqrt{t^2 + t^4}$, so we integrate from $t = 0$ to $t = 1$ and get:

$$L = \int_0^1 \sqrt{t^2 + t^4}\,dt = \int_0^1 t\sqrt{1 + t^2}\,dt = \int_1^2 \frac{1}{2}\sqrt{u}\,du = \frac{2\sqrt{2} - 1}{3}$$

Extra Credit: (15 points) Given two points on the sphere with spherical coordinates $\rho = 1$, $\phi = \frac{\pi}{2}$, $\theta = 0$, and $\rho = 1$, $\phi = \frac{\pi}{4}$, $\theta = \frac{\pi}{4}$. Find the shortest distance along the sphere between these two points.

SOLUTION: These two points determine a great circle on the sphere. Thus the shortest distance between them along the surface of the sphere lies along the great circle. We need only find the angle between these vectors and then figure out the length of the arc subtended by them.

Convert our points into cartesian coordinates. We get:

$$(1, 0, 0) \quad \text{and} \quad \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$$

Find the angle between them:

$$\psi = \cos^{-1}\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

On a circle of raidius 1 the length of the arc subtended by this angle is $\frac{\pi}{3}$; which is the distance between the two points along the surface of the sphere.