

Practice Problems for the Final, Fall 2006

1. Find the plane that contains the lines $L_1 : x = 1+t, y = 1-t, z = -t$ and $L_2 : x = 1+3t, y = -3t, z = -3-3t$.
2. Find and classify the critical points of the function $f(x, y) = 4x^2e^y - 2x^4 - e^{4y}$.
3. Find the absolute maximum of the function $f(x, y, z) = x^2yz$ on the triangle cut by the plane $x + y + z = 12$ and the first octant.
4. A thin wire lies along the curve $\mathbf{r}(t), 0 \leq t \leq 1$. The wire density δ (its mass per unit length) increases linearly along the wire and is $\delta(\mathbf{r}) = s + 1$, where s is the arc length parameter, and $s = 0$ corresponds to the $t = 0$ endpoint of the wire. Determine the wire mass if $|\mathbf{v}(t)| = t^2 + 1$.
5. Consider the gravitational field:

$$\mathbf{F} = -GMm \frac{(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{(x^2 + y^2 + z^2)^{3/2}},$$

where G, M, m are constants.

- (a) Show that the field is conservative. (Hint: Set $r = \sqrt{x^2 + y^2 + z^2}$, and use that $\frac{\partial r}{\partial x} = \frac{x}{r}$.)
 - (b) Find a potential function $f(x, y, z)$ for F . (Hint: Try f of the form C/r where C is a constant.)
 - (c) Let P and Q be points at distances s and t from the origin. Find the work done by the gravitational field \mathbf{F} in moving a particle from P to Q , i.e. evaluate the integral $\int_P^Q \mathbf{F} \cdot d\mathbf{r}$.
6. Let \mathbf{F} be the vector field

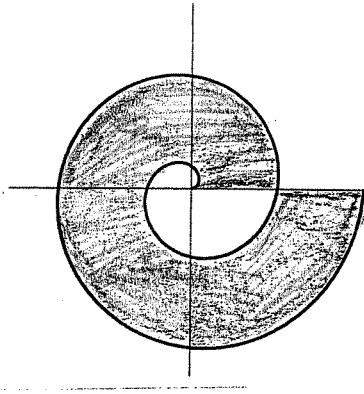
$$\mathbf{F}(x, y, z) = (x^2 + y + zy^3)\mathbf{i} + ax(y + 1)\mathbf{j} + (x^3 + y^2 + xyz^3)\mathbf{k}$$

where a is a real number. Find the parameters a such that the flux of $\text{curl } \mathbf{F}$ across the unit upper hemisphere is zero.

7. Let

$$f(x, y, z) = (x^2 + y^2 + z^2)^a$$

- (a) Compute $\text{div}(\text{grad}(f))$.
 - (b) For which value(s) of a is $\text{div}(\text{grad}(f)) = 0$?
8. (a) Graph the part of the spiral $r = \theta$ for $0 \leq \theta \leq 4\pi$.
 - (b) Calculate the area of the shaded region, shown in the picture.



Final Exam, Fall 2005

1. Let $f(x, y) = \sqrt{1 - \frac{x^2}{2} - \frac{y^2}{16}}$.

a) Find the equation of the tangent plane to the graph of f above the point $(1, 2)$.

b) Using linear approximation, estimate the value of f at the point $(0.8, 2.4)$.

2. Let $\vec{F} = -y\vec{i} + x\vec{j}$ be a vector field in the plane. Let C be a simple closed curve enclosing a domain D .

a) If C is parametrized in a counterclockwise direction, show that

$$\frac{1}{2} \int_C \vec{F} \cdot \vec{T} ds = \text{Area of } D.$$

b) Use Part (a) to compute the area of the ellipse parameterized by $\vec{r}(t) = (a \cos t)\vec{i} + (b \sin t)\vec{j}$, where a and b are constants, and $0 \leq t \leq 2\pi$.

3. Find the area of the region that lies outside of the circle $r = 1$ and inside the circle $r = 2 \sin \theta$.

4. Find the volume of the smaller region cut from the solid sphere $\rho \leq 2$ by the plane $z = 1$.

5. Consider the region (in the plane) given by $x^2 + y^2 - xy \leq 1$ (this is inside of a “tilted ellipse”). Let R be the part of this region that lies on or above the line $y = 1$. Let $f(x, y) = x^2 - xy$. Find the maximum and minimum values of f over R .

6. Consider the (upper) hemisphere defined by $x^2 + y^2 + z^2 = 1$ where $z \geq 0$. Let S be the part of this hemisphere that lies within the cylinder $(x - \frac{1}{2})^2 + y^2 \leq \frac{1}{4}$. Find the average distance of a point on S to the xy -plane.

7. Let a be a constant and consider the vector field, defined by

$$\vec{F} = (e^x \cos y)\vec{i} + (-e^x \sin y + z^2 + (1 - a^2)z)\vec{j} + (2a^2yz)\vec{k}.$$

a) Find all values of the parameter a that make \vec{F} conservative.

b) For $a = 1$, \vec{F} is conservative. In this case, find a potential function f for \vec{F} .

c) For $a = 1$, evaluate the work done by \vec{F} along any curve from $(0, 0, 7)$ to $(1, \frac{\pi}{2}, 2)$.

8. Define S to be the part of the cylinder $x^2 + y^2 = 4$ that lies between the planes $z = 0$ and $z = 3$. Evaluate the flux of the vector field $\vec{F} = x\vec{i} + y\vec{j} + e^z\vec{k}$ across S , in the outward direction.

9. The statement of Stokes' Theorem involves a surface integral and a line integral. Let S be the part of the surface $x^2 + y^2 + 5z = 1$ that lies above the xy -plane. Consider the vector field $\vec{F} = xz\vec{i} + yz\vec{j} + (x^2 + y^2)\vec{k}$. Orienting S using the upward pointing normal, evaluate both integrals in Stokes' Theorem (using the surface S and vector field \vec{F}) and confirm they are indeed equal.

10. Consider the region (in space) bounded above by $z = e^{1-x^2-y^2}$ and bounded below by the plane $z = 1$. Define S to be the boundary of this region. Let

$$\vec{F} = 2x\vec{i} + y\vec{j} + (2 - 2z)\vec{k}.$$

Compute the flux integral $\int_S \vec{F} \cdot \vec{n} d\sigma$, where \vec{n} is the outward pointing normal field.