Math 192 – Practice Final Exam – Fall 2004

1. Consider the integral $\int_0^1 \int_{y^2}^{y^{1/3}} xy^2 dxdy$.

   (a) Evaluate the integral.
   (b) Sketch the region of integration.
   (c) Reverse the order of integration.

2. Set up a triple integral (or a sum of triple integrals) to compute the volume of the region bounded below by the plane $z = 0$, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$. Use cylindrical coordinates and the order of integration $drdzd\theta$. Sketch the region. Do not evaluate the integral.

3. Consider the integral $\int\int_D\int \cos^2(\phi)\, dV$, where D is the region bounded below by the plane $z = 0$, above by the cone $\phi = \frac{\pi}{3}$, and on the sides by the sphere $\rho = 2$. (Here ϕ and ρ are spherical coordinates.)

   (a) Set up the triple integral in spherical coordinates using the order of integration $d\rho d\phi d\theta$.
   (b) Evaluate the integrals.

4. The following two lines are given: $L1 : x = 1 - 2t,\ y = 4t,\ z = -8 - 3t$ and $L2 : x = 4 + s,\ y = -1 - 3s,\ z = -6 + 2s$.

   (a) Do the lines intersect? If yes, find the intersection point.
   (b) Find the plane that contains both lines.

5. The acceleration vector of a particle in three dimensional space is $\mathbf{a}(t) = -\mathbf{i} - \mathbf{j} - 6t\mathbf{k}$. The velocity of the particle at time $t = 0$ is $\mathbf{v}(0) = \mathbf{0}$. The particle is at the point $(10, 2, 5)$ at time $t = 0$. Calculate the position vector $\mathbf{r}(t)$ of the particle as a function of t. Where (at what point) is the particle at time $t = 2$?

6. Consider the function $f(x, y) = 2xy - x^2 - 2y^2 + 3x + 4$.

   (a) Find all critical points. Find the local maxima and local minima, and all points where they occur.
   (b) Find the tangent plane to the graph surface $z = f(x, y)$ at the point $(1, 0, 6)$.

7. Consider the function $f(x, y) = xy^2 + y\cos(x - 1)$.

   (a) Find the linearization at the point $(1, 2)$.
   (b) Find an upper bound for the magnitude $|E|$ of the error in the approximation of f by the linearization over the rectangle $|x - 1| \leq 0.1,\ |y - 2| \leq 0.1$.

8. A plane cuts a sphere of radius R. The plane is at a distance h from the center of the sphere. (We assume that $R > h > 0$.) Set up and evaluate a triple integral computing the volume of the region that contains the center of the sphere, and is bounded by the plane and by the sphere.

9. Find all local minima, local maxima, and saddle points of $z = x^2 + y^2 - xy - x$.

10. Find the area of one leaf of the rose $r = \sin(3\theta)$.

11. Consider the integral $\int_{-1}^{1}\int_{x^2}^{\sqrt{2-x^2}} ydydx$.

(a) (5 points) Describe or sketch the region of integration.
(b) (5 points) Evaluate the integral.
(c) (10 points) Reverse the order of integration.

12. Set up triple integrals to compute the volumes of the given regions. Do not evaluate the integrals.

(a) (8 points) Let D be the region bounded above by the paraboloid $z = 8 - x^2 - y^2$ and below by the paraboloid $z = x^2 + y^2$. Use cartesian coordinates and the order of integration $dydzdx$.
(b) (9 points) Let D be the region bounded below by the xy-plane, on the sides by the sphere $\rho = \sqrt{2}$, and above by the cone $z = \sqrt{x^2 + y^2}$. Use cylindrical coordinates and the order of integration $dzdrd\theta$.
(c) (8 points) Let D be the region bounded above by the sphere of radius 2 centered at $(0,0,0)$ and below by the cone $z = \sqrt{x^2 + y^2}$. Use spherical coordinates and the order of integration $d\rho d\phi d\theta$.

13. (20 points) Calculate the volume of the cylinder $x^2 + y^2 \leq 1$ between the planes $z = x$ and $z = 2 + 2x$.

14. (20 points) Find the mass of a spherical ball of radius R with density

$$\delta(x, y, z) = k(\text{distance from center of ball to } (x, y, z)).$$

15. (20 points) The following two non-intersecting lines are given: $L1 : x = 2 + 3t,\ y = 1 - 4t,\ z = 3 - 2t$ and $L2 : x = 3 + 3s,\ y = -2 + s,\ z = 5 - s$. Find two parallel planes, such that $L1$ lies in one of the planes and $L2$ lies in the other plane.

16. Consider the function $f(s,t) = \int_s^t (6 - x - x^2)dx$ defined in the domain $s \leq t$ (that is, the domain is the unbounded half-plane $s \leq t$).

(a) (10 points) Find the critical points of the function. Find the local maxima and local minima, and the points where they occur.
(b) (15 points) Does the absolute maximum exist? Explain.

17. (a) (12 points) Calculate the directional derivative of the function $f(x, y) = e^{x^2+y^2}$ at the point $(2, 1)$ in the direction of $\mathbf{i} - \mathbf{j}$. Find an equation of the tangent plane to the surface $z = e^{x^2+y^2}$ at the point $(2, 1, e^5)$.

(b) (13 points) If $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ and $f(x, y) = \ln\sqrt{x^2 + y^2}$, write an expression for ∇f in terms of $\mathbf{r}$.

18. (25 points) Wire of total length 1000 cm is formed into a flexible coil in the shape of the circular helix $x = 3\cos t,\ y = 3\sin t,\ z = bt$, where b is a constant. There are 10 turns to each centimeter of height and the radius of the helix is 3 cm. How tall is the coil?

19. Let $f(x, y) = x^2 \sin y$.

(a) (12 points) Find the linearization for $f(x, y)$ about the point $(1, \pi/2)$ and use your answer to approximate $f(0.9, \pi/2 + 0.2)$.
(b) (12 points) Find an upper bound for the error in your estimate from part a.

20. A particle's velocity is given by $\mathbf{v}(t) = t\mathbf{i} + t^2\mathbf{j}$.

(a) (12 points) If the particle's position at time $t = 0$ is (2,1), what is the particle's position at time $t = 1$?

(b) (12 points) How far does the particle travel from time $t = 0$ to $t = 1$?

Extra Credit: (15 points) Given two points on the sphere with spherical coordinates $\rho = 1, \phi = \frac{\pi}{2}, \theta = 0$, and $\rho = 1, \phi = \frac{\pi}{4}, \theta = \frac{\pi}{4}$. Find the shortest distance along the sphere between these two points.

Sample problems on Chapter 14

Problem 1: Find the surface area of the torus T (shaped like a bagel) with surface described in parameterized form by

$$x = (3 + \cos t)\cos\theta, \quad y = (3 + \cos t)\sin\theta, \quad z = \sin t,$$

where $0 \leq t, \theta \leq 2\pi$.

Problem 2: Evaluate the line integral of $\cos x + \sin y$ on the curve $C : y = x, 0 \leq x \leq \pi/2$.

Problem 3: Evaluate the integral of y over the surface

$$S : z = \cosh x, \qquad 0 \leq x \leq 1, \quad 0 \leq y \leq 1.$$

Problem 4: Find the work done by $\mathbf{F} = (xy)\mathbf{i} + (y)\mathbf{j} + (yz)\mathbf{k}$ over the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq 1$.

Problem 5: Let $\mathbf{F} = (2y\sin z)\mathbf{j} + (y^2\cos z)\mathbf{k}$ represent a force field and consider a curved line C described by $\mathbf{r}(t) = t\cos t^2\mathbf{i} + t\sin t^2\mathbf{j} + t^2\mathbf{k}$, $0 \leq t \leq \sqrt{\pi/2}$. Calculate the work integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ by any legitimate means.

Problem 6: Given the vector field $\mathbf{F} = (y + 3)\mathbf{i} + (x - z)\mathbf{j} + (\sin z - y)\mathbf{k}$ and the path γ described by two connected straight line segments from $(0, 0, 0)$ to $(1, 2, 1)$ to $(0, 0, 1)$

a) Find a parameterization for a straight line segment from $(0, 0, 0)$ to $(1, 2, 1)$.

b) is the vector field F a gradient vector field? (You must support your answer).

c) Find the work done in the field $\mathbf{F}$ by moving from $(0, 0, 0)$ to $(0, 0, 1)$ along γ.

Problem 7: a) Find a function f whose gradient field is $(y + \ln(1 + y^2))\mathbf{i} + (x + \frac{2xy}{1+y^2})\mathbf{j}$.

b) Compute the line integral $\int_C (y + \ln(1 + y^2))dx + (x + \frac{2xy}{1+y^2})dy$ where C is the path $\mathbf{r}(t) = t^2\sin(\frac{\pi t}{2})\mathbf{i} + e^{t^2-1}\tan(\frac{\pi t}{4})\mathbf{j}$, $0 \leq t \leq 1$.

Problem 8: Consider the vector field $\mathbf{F} = x\mathbf{i} + (\cos z)\mathbf{j} + (1 - y\sin z)\mathbf{k}$.

a) Show that this field is conservative and calculate a potential function for it.

b) For this vector field, calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve

$$x(t) = t^2, y(t) = t^3, z(t) = t^4, 0 \leq t \leq 1.$$

Problem 9: For the vector field $\mathbf{F} = (yz)\mathbf{i} + (xz)\mathbf{j} + (xy)\mathbf{k}$ compute the flow along the curve $\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j} + t^5\mathbf{k}$, $0 \leq t \leq 1$.

Problem 10: Find a potential function for the vector field

$$\mathbf{F} = (1 + \sin y\cos z)\mathbf{i} + (2y + x\cos y\cos z)\mathbf{j} + (3z^2 - x\sin y\sin z)\mathbf{k}.$$

Problem 11: Using Green's theorem, find the outward flux of $\mathbf{F} = xy\mathbf{i} + x\mathbf{j}$ across the closed curve C which is the boundary of a triangle T with vertices at $(0,0)$, $(7,0)$ and $(0,7)$. Sketch the triangle.

Problem 12: Using Green's theorem, find the counterclockwise circulation of $\mathbf{F} = (y^2 - x^2)\mathbf{i} + (x^2 + y^2)\mathbf{j}$ on the triangle bounded by $y = 0$, $x = 3$ and $y = x$. Sketch the triangle.

Problem 13: a) Use Green's theorem to compute the outward flux of the vector field $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j}$ across the boundary curve of the region D in the first quadrant lying between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Compute the same integral directly.
b) Also use the Green's theorem to compute the counterclockwise circulation of F around the boundary curve of this region.

Problem 14: Let C be the curve $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$.
a) Find the area bounded by C as follows: let $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ and calculate the $\oint_C \mathbf{F} \cdot d\mathbf{r}$. Use an appropriate form of Green's theorem to get the answer.
b) Find the area without using calculus. Check that both methods give the same answer.

Problem 15: Consider the vector field $\mathbf{F} = (x \sin x^2)\mathbf{i} + (x^2 z)\mathbf{j} + (z^5 + x - y)\mathbf{k}$ and the curve C which is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + z = 1$ traveled counterclockwise as seen from the point $(1,0,1)$.

a) Sketch the curve C, labeling axes and important distances, and indicate the direction of travel along the curve.
b) Calculate the circulation of F along C. (Hint: Stokes's theorem can be helpful).

Problem 16: Consider a cone $S : z = 2 - \frac{1}{2}\sqrt{x^2 + y^2}, 1 \leq z \leq 2$. Calculate the flux $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma$ of the curl of the vector field $\mathbf{F} = yx^2\mathbf{i} - xy^2\mathbf{j} + \mathbf{k}$ across S in the direction of the outward unit normal vector $\mathbf{n}$ using either an appropriate line integral or a surface integral (your choice).

Problem 17: For the vector field $\mathbf{F} = (xy^2)\mathbf{i} + (yz^2)\mathbf{j} + (zx^2)\mathbf{k}$ compute the outward flux across the boundary surface of the region D inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{x^2 + y^2}$.

Problem 18: Use the divergence theorem to find the outward flux of the field $\mathbf{F} = 2xz\mathbf{i} - xy\mathbf{j} - z^2\mathbf{k}$ across the boundary of the solid wedge D cut from the first octant by the plane $y + z = 4$ and the elliptical cylinder $4x^2 + y^2 = 16$.

Problem 19: Calculate the outward flux through the upper unit hemisphere $\{z \geq 0, x^2 + y^2 + z^2 \leq 1\}$, including the equatorial plane, for the vector field $\mathbf{F} = xy^2\mathbf{i} + x^2y\mathbf{j} + \frac{z^3}{3}\mathbf{k}$.

Problem 20: Use the divergence theorem to calculate the flux of the vector field $\mathbf{F} = (x^3)\mathbf{i} + (y^3)\mathbf{j} + (z^3)\mathbf{k}$ outward across the unit sphere centered at the origin.