Math 192, Final Exam, May 9, 2008

So put me on a highway, And show me a sign, And take it to the limit one more time
- the Eagles

This test is worth 150 *points. You are allowed one sheet of paper. You are NOT allowed calculators or the text. SHOW ALL WORK!*

1) Let V_0 be the region bound by $x^2 + y^2 \leq 1$, $z \geq 0$ and $x + z \leq 2$.
a) (10 points) The points $(0, 1, 1)$ and $(0, 0, 2)$ are on the surface of V_0. Find the equation of the tangent planes to V_0 at these two points. Find the equation of the line of intersection of these two planes.
b) (7 points) Find the volume of V_0.

2) Let $f(x, y) = x^2 + 1 + (\cos y)(\cos x)$
a) (7 points) Find the average value of $f(x, y)$ on the rectangle with vertices $(0, 0)$, $(a, 0)$, (a, b) and $(0, b)$.
b) (7 points) Denote the average value in part a) above by $A(a, b)$. Find $\lim\limits_{(a,b)\to(0,0)} A(a, b)$ where $a, b > 0$ as we take the limit.

3) Let S be the portion of the plane $y = \sqrt{3}x$ that lies inside the sphere $x^2 + y^2 + z^2 = 9$.
a) (7 points) Parametrize S.
b) (7 points) Find $\displaystyle\int\int_S x^2\, d\sigma$.

4) The questions below are true/false. Either write the entire word TRUE, the entire word FALSE or leave it blank. No work is required for this problem. A correct answer is worth 3 points. A blank answer is worth 0 points. A wrong answer is worth -4 points, so **don't guess!**
a) $(+3/-4$ points) For any vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in three dimensional space $\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{v} \cdot (\vec{u} \times \vec{w})$.
b) $(+3/-4$ points) Consider the function $f(x, y, z)$. Assume all its partial derivatives exist and are continuous. Suppose $f(1, 2, -3) = \pi$. Then $\vec{\nabla} f(1, 2, 3)$ is tangent to the level surface $f(x, y, z) = \pi$.
c) $(+3/-4$ points) Let $f(x, y, z)$ be a function. Assume all its partial derivatives exist and are continuous. Consider the constraint equation $g(x, y, z) = x + 2y - 4z = 0$. f *must* have constrained extrema on the plane $g(x, y, z) = 0$.
d) $(+3/-4$ points) Conservative vector fields always have divergence equal to zero.
e) $(+3/-4$ points) If $\vec{F}$ and $\vec{G}$ are conservative vector fields then $\vec{H} = \vec{F} + \vec{G}$ is conservative.

5) a) (7 points) Give an example, with justification, of a scalar function $f(x, y, z)$ whose gradient is *never* $\vec{0}$, that is $\vec{\nabla} f(x, y, z) \neq \vec{0}$ for any (x, y, z).
b) (7 points) Give an example,with justification, of a three dimensional vector field $\vec{F}$ that is defined at all points in three dimensional space whose divergence is *never* 0, that is $(\vec{\nabla} \cdot \vec{F})(x, y, z) \neq 0$ for any (x, y, z).
c) (7 points) Give an example, with justification, of a three dimensional vector field $\vec{F}$ that is defined at all points in three dimensional space whose curl is *never* $\vec{0}$, that is $(\vec{\nabla} \times \vec{F})(x, y, z) \neq \vec{0}$ for any

(x, y, z).

6) Let $A = (0,0,0)$, $B = (1,0,0)$, $C = (1,1,4)$ and $D = (0,1,4)$. Let $\vec{F} = y\vec{\imath} - z\vec{\jmath} + 2x\vec{k}$.
a) (7 points) Find the work done by $\vec{F}$ along the line segment starting at A and ending at D.
b) (7 points) Find $\vec{\nabla} \times \vec{F}$.
c) (7 points) Find the work done by $\vec{F}$ around the rectangle $ABCD$.

7) a) (8 points) Calculate the counterclockwise circulation of $\vec{F} = -y\vec{\imath} + x\vec{\jmath}$ around a simple curve C in the plane. You are given that the area of the region R enclosed by C is $\dfrac{a(a-1)b(b-1)}{2}$ where a and b are constants.
b) (8 points) Find the maximum circulation from (a) where $0 \le a \le 1$ and $0 \le b \le 1$.

8) a) (8 points) For a scalar function $w(x, y, z)$ all of whose partial derivatives to all orders exist, we use the symbol $\nabla^2 w$ to denote $\vec{\nabla} \cdot \vec{\nabla} w$. Show $\vec{\nabla} \cdot \vec{\nabla} w = w_{xx} + w_{yy} + w_{zz}$.
b) (8 points) Let $g(x, y, z)$ and $h(x, y, z)$ be functions all of whose partial derivatives to all orders exist. Set $\vec{F} = g\vec{\nabla} h$. Show $\vec{\nabla} \cdot \vec{F} = g\nabla^2 h + (\vec{\nabla} g) \cdot (\vec{\nabla} h)$.

9) Let $\vec{F} = y\vec{\imath} + (x - 2y)\vec{\jmath} + z\vec{k}$ and consider the cylinder $x^2 + y^2 = R^2$.
a) (8 points) Parametrize the surface of that part of the cylinder that satisfies $-R \le z \le 2R$. (Don't do the top or bottom). At what points in the domain of your parametrization is area 'stretched the most' and 'stretched the least'?
b) (8 points) Find the flux of $\vec{F}$ through that part of the cylinder described in part a).