Math 1910, Prelim 2
October 29, 2009
Solutions

1) a) Letting $u = \ln(x)$ we have $du = \frac{dx}{x}$, $x = 1 \Rightarrow u = 0$, $x = 2 \Rightarrow u = \ln(2)$. Thus

$$\int_1^2 \frac{\log_2(x)}{x}\, dx = \frac{1}{\ln(2)} \int_1^2 \frac{\ln(x)}{x}\, dx = \int_0^{\ln(2)} u\, du = \frac{1}{\ln(2)} \left[\frac{1}{2}u^2\right]_0^{\ln(2)} = \frac{\ln(2)}{2}$$

b) Letting $u = 3 - e^x$ we have $du = -e^x dx$, $x = 0 \Rightarrow u = 2$, $x = \ln(2) \Rightarrow u = 1$. Therefore

$$\int_0^{\ln(2)} \frac{e^x}{(3 - e^x)^2}\, dx = \int_2^1 \frac{-du}{u^2} = \left[\frac{1}{u}\right]_2^1 = \frac{1}{1} - \frac{1}{2} = \frac{1}{2}$$

c) Letting $u = e^{2x}$ we have $e^{4x} = u^2$ and $du = 2e^{2x} dx$. So

$$\int \frac{e^{2x}}{1 + e^{4x}}\, dx = \frac{1}{2} \int \frac{du}{1 + u^2} = \frac{1}{2} \tan^{-1}(u) + C = \frac{1}{2} \tan^{-1}(e^{2x}) + C$$

d) Use integration by parts.

$$\begin{aligned}
\int \ln^2(x)\ dx &= \int 1 \cdot \ln^2(x)\ dx \\
&= x \ln^2(x) - \int x \cdot 2 \ln(x) \frac{1}{x}\ dx \\
&= x \ln^2(x) - 2 \int \ln(x)\ dx \\
&= x \ln^2(x) - 2 \int 1 \cdot \ln(x)\ dx \\
&= x \ln^2(x) - 2(x \ln(x) - \int x \cdot \frac{1}{x}\ dx) \\
&= x \ln^2(x) - 2x \ln(x) + 2 \int dx \\
&= x \ln^2(x) - 2x \ln(x) + 2x + C
\end{aligned}$$

e) Write $\frac{x+1}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$. Then $x + 1 = A(x^2 + 1) + (Bx + C)x = (A + B)x^2 + Cx + A$. This shows that $A + B = 0$, $C = 1$, $A = 1$. Thus we must have $A = 1$, $B = -1$ and $C = 1$. Therefore

$$\int \frac{1}{x^3 + x}\, dx = \int (\frac{1}{x} - \frac{x}{x^2 + 1} + \frac{1}{x^2 + 1})\, dx = \ln|x| - \frac{1}{2} \ln|x^2 + 1| + \tan^{-1}(x) + C'$$

2) a) Since $\lim_{x\to\infty} \frac{1-e^{-2x}}{2} = \frac{1-0}{2} = \frac{1}{2}$ we have that eventually $\frac{\sinh(x)}{e^x} = \frac{\frac{e^x - e^{-x}}{2}}{e^x} = \frac{1-e^{-2x}}{2} \le 1$. Thus $\sinh(x) = O(e^x)$. However since $\frac{\sinh(x)}{1} = \sinh(x) \to \infty$ as $x \to \infty$ then $\sinh(x)$ can't be $O(1)$.

b) Since $\lim\limits_{x\to\infty}\tanh(x) = \lim\limits_{x\to\infty}\frac{\sinh(x)}{\cosh(x)} = \lim\limits_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}} = \lim\limits_{x\to\infty}\frac{1-e^{-2x}}{1+e^{-2x}} = \frac{1-0}{1+0} = 1$ we get that eventually $\frac{\tanh(x)}{1} = \tanh(x) \le 2$. Thus $\tanh(x) = O(1)$. Since eventually $\frac{\tanh(x)}{e^x} \le \tanh(x) \le 2$ we also have $\tanh(x) = O(e^x)$.

c) Since $|\tan^{-1}(x)| \le \frac{\pi}{2}$ then it is clear that $\frac{\tan^{-1}(x)}{1} \le 2$ which shows $\tan^{-1}(x) = O(1)$. Also $\frac{\tan^{-1}(x)}{e^x} \le \tan^{-1}(x) \le 2$ which shows $\tan^{-1}(x) = O(e^x)$.

d) Since $\frac{e^x(1+\sin(x))}{e^x} = 1+\sin(x) \le 2$ then $e^x(1+\sin(x)) = O(e^x)$. However if we take $x_n = \frac{\pi}{2} + n\cdot 2\pi$ then $e^{x_n}(1+\sin(x_n)) = 2e^{\frac{\pi}{2}+n\cdot 2\pi} \to \infty$ as $n \to \infty$ which shows that our function can become as big as we want it. Thus it can't be $O(1)$.

3) a) $\dfrac{dy}{dx} = \dfrac{d}{dx}(e^{x\ln(x)}) = e^{x\ln(x)}(\ln(x) + x\cdot\dfrac{1}{x}) = x^x(\ln(x)+1)$

b) Note that $\frac{dy}{dx} = 0 \Rightarrow \ln(x)+1 = 0 \Rightarrow x = e^{-1}$. By looking at how the sign of $\frac{dy}{dx}$ changes with x we can see that y is decreasing on $]0, e^{-1}[$ and increasing on $]e^{-1}, \infty[$.

c) Let's find first the limit of $\ln(x^x)$.

$$\begin{aligned}\lim_{x\to 0^+}\ln(x^x) &= \lim_{x\to 0^+} x\ln(x)\\ &= \lim_{x\to 0^+}\frac{\ln(x)}{1/x}\\ &= \lim_{x\to 0^+}\frac{1/x}{-1/x^2}\\ &= \lim_{x\to 0^+} -x\\ &= 0\end{aligned}$$

Thus $\lim\limits_{x\to 0^+} x^x = e^0 = 1$.

4) a) We show that $H(t)$ is one to one by showing $H'(t) > 0$ so that $H(t)$ is increasing.

$$H'(t) = \frac{100}{\pi}\frac{1}{1+t^2} + 100\cdot\frac{(t+1)-t}{(t+1)^2} = \frac{100}{\pi}\frac{1}{1+t^2} + \frac{100}{(t+1)^2}$$

For $t \ge 0$ both terms in $H'(t)$ are positive, so $H'(t) > 0$.

b) Since $H^{-1}(75C) = 1$ we have

$$(H^{-1})'(75) = \frac{1}{H'(H^{-1}(75C))} = \frac{1}{H'(1)} = \frac{1}{50/\pi + 25} = \frac{\pi}{50+25\pi}$$

5) a) $y(t) = y_0 e^{-2t}$

b) Since $\lim\limits_{t\to\infty} e^{-2t} = 0$ we have $\lim\limits_{t\to\infty} y(t) = 0$ regardless of the initial size y_0.

c) Solve $\frac{1}{4}y_0 = y_0 e^{-2t}$ for t and obtain $\ln(\frac{1}{4}) = -2t$ and thus $t = \ln(2)$. This answer depends on $y_0 \ne 0$. If $y_0 = 0$ then the population is already quarter of the starting size, $\frac{1}{4}\cdot 0 = 0$, so then the answer is $t = 0$.