HW7 Solutions

7.3 The Exponential Function

4.

$$a)\ \ln e^{\sec(\theta)} = (\sec(\theta))(\ln e) = \sec\theta$$

$$b)\ \ln e^{(e^x)} = (e^x)(\ln e) = e^x$$

$$c)\ \ln e^{2\ln x} = \ln e^{\ln x^2} = \ln x^2 = 2\ln x$$

30.

$$y = \ln 2e^{-t}\sin t = \ln 2 + \ln e^{-t} + \ln \sin t = \ln 2 - t + \ln\sin t \to \frac{dy}{dx} = -1 + \frac{1}{\sin t}\frac{d}{dt}(\sin t) = -1 + \frac{\cos t}{\sin t} = \frac{\cos t - \sin t}{\sin t}$$

36.

$$y = \int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t\, dt \to y' = (\ln e^{2x}) * \frac{d}{dx}(e^{2x}) - (\ln e^{4\sqrt{x}}) * \frac{d}{dx}(e^{4\sqrt{x}}) = 4xe^{2x} - 8e^{4\sqrt{x}}$$

54. Let

$$u = -x^{-2} \to du = 2x^{-3}dx \to \frac{1}{2}du = x^{-3}dx;$$

$$\int \frac{e^{-1/x^2}}{x^3}\, dx = \int e^{-x^{-2}} * x^{-3}\, dx = \frac{1}{2}\int e^u\, du = \frac{1}{2}e^u + C = \frac{1}{2}e^{-x^{-2}} + C$$

68. The function $f(x) = 2e^{\sin x/2}$ has a maximum whenever $\sin\frac{x}{2} = 1$ and a minimum whenever $\sin\frac{x}{2} = -1$. Therefore, the maximums occur at $x = \pi + 2k(2\pi)$ and the minimums occur at $x = 3\pi + 2k(2\pi)$, where k is any integer. The maximum is $2e \approx 5.43656$ and the minimum is $\frac{2}{e} \approx 0.73576$.

7.4 a^x and $\log_a x$

8.

$$8^{\log_8 3} - e^{\ln 5} = x^2 - 7^{\log_7 3x} \to 3 - 5 = x^2 - 3x \to 0 = x^2 - 3x + 2 = (x-1)(x-2) \to x = 1, x = 2$$

22.

$$y = 5^{-\cos 2t} \to \frac{dy}{dt} = (5^{-\cos 2t}\ln 5)(\sin 2t)(2) = (2\sin 2t)(5^{-\cos 2t})(\ln 5)$$

46.

$$y = (\ln x)^{\ln x} \to \ln y = (\ln x)\ln(\ln x) \to \frac{y'}{y} = \frac{1}{x}\ln(\ln x) + (\ln x)\frac{1}{\ln x}\frac{d}{dx}(\ln x) = \frac{\ln(\ln x)}{x} + \frac{1}{x} \to y' = \frac{\ln(\ln x) + 1}{x}(\ln x)^{\ln x}$$

67.

$$\int_0^9 \frac{2\log_1 0(x+1)}{x+1}\, dx = \frac{2}{\ln 10}\int_0^9 \ln(x+1)\frac{1}{x+1}\, dx = \frac{2}{\ln 10}\frac{(\ln(x+1))^2}{2} = \ln 10$$

7.5 Exponential Growth and Decay

2.

a) $\frac{dp}{dh} = kp \to p = p_0 e^{kh}$ where $p_0 = 1013e^{20k} \to k = \frac{\ln 90 - \ln 1013}{20} \approx -0.121$

b) $p = 1013e^{-6.05} \approx 2.389$ millibars
c) $y = y_0 e^{-0.121h} \rightarrow -0.121h = \ln \frac{900}{1013} \rightarrow h = \frac{\ln 1013 - \ln 900}{0.121} \approx 0.977$ km

6. $V(t) = V_0 e^{-t/40} \rightarrow 0.1V_0 = V_0 e^{-t/40}$ when the voltage is 10 % of its original value $\rightarrow t = -40 \ln 0.1 \approx 92.1$ sec

10.
a) There are (60)(60)(24)(365) = 31,536,000 seconds in a year. Thus, assuming exponential growth, $P = 257,313,431e^{kt}$ and

$$257,313,431e^{14k/31,536,000} \rightarrow \ln \frac{257,313,432}{257,313,431} = \frac{14k}{31,536,000} \rightarrow k \approx 0.0087542$$

b)

$$P = 257,313,431e^{0.0087542*15} \approx 293,420,847$$

Answers will vary considerably with the number of decimal places retained.

18.
$A = A_0 e^{kt}$ and $\frac{1}{2}A_0 = A_0 e^{139k} \rightarrow \frac{1}{2} = e^{139k} \rightarrow k = \frac{\ln 0.5}{139} \approx -0.00499$; then $0.05A_0 = A_0 e^{-0.00499t} \rightarrow t = \frac{\ln 0.05}{-0.00499} \approx 600$ days

25.
From example 5, the half-life of carbon-14 is 5700 years so

$$\frac{1}{2}c_0 = c_0 e^{-5700k} \rightarrow k = \frac{\ln 2}{5700} \approx 0.0001216 \rightarrow c = c_0 e^{-0.0001216t} \rightarrow (0.445)c_0 = c_0 e^{-0.0001216t} \rightarrow t = \frac{\ln 0.445}{-0.0001216} \approx 6659 years$$

7.8 Hyperbolic Functions

10.

$$\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x) = \ln(\cosh^2 x - \sinh^2 x) = \ln 1 = 0$$

12.

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{1}{4}[e^x + e^{-x} + e^x - e^{-x}][e^x + e^{-x} - e^x - e^{-x}] = \frac{1}{4}(2e^x)(2e^{-x}) = \frac{1}{4}(4) = 1$$

77.
a)

$$v = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{gk}{m}}t\right) \rightarrow \frac{dv}{dt} = \sqrt{\frac{mg}{k}}\left[sech^2\left(\sqrt{\frac{gk}{m}}t\right)\right]\left(\sqrt{\frac{gk}{m}}\right) = gsech^2\left(\sqrt{\frac{gk}{m}}t\right)$$

Thus

$$m\frac{dv}{dt} = mgsech^2\left(\sqrt{\frac{gk}{m}}t\right) = mg\left(1 - \tanh^2\left(\sqrt{\frac{gk}{m}}t\right)\right) = mg - kv^2$$

b)

$$\lim_{t\to\infty} v = \lim_{t\to\infty} \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}}t\right) = \sqrt{\frac{mg}{k}} \lim_{t\to\infty} \tanh\left(\sqrt{\frac{kg}{m}}t\right) = \sqrt{\frac{mg}{k}}(1) = \sqrt{\frac{mg}{k}}$$

c)

$$\sqrt{\frac{160}{0.005}} = \sqrt{\frac{160,000}{5}} = \frac{400}{\sqrt{5}} = 80\sqrt{5} \approx 178.89$$

87.

a)

$$y = \frac{H}{\omega}\cosh\left(\frac{\omega}{H}x\right) \rightarrow \tan\phi = \frac{dy}{dx} = \frac{H}{\omega}\left[\frac{\omega}{H}\sinh\left(\frac{\omega}{H}x\right)\right] = \sinh\left(\frac{\omega}{H}\right)$$

b) The tension at P is given by

$$T\cos\phi = H \rightarrow T = H\sec\phi = H\sqrt{1+\tan^2\phi} = H\sqrt{1+\left(\sinh\left(\frac{\omega}{H}x\right)\right)^2} = H\cosh\left(\frac{\omega}{H}x\right) = (\omega)(y)$$