September 29, 2009

# PRELIM 1 V2 Solution V5 Math 1710

## Directions:

- No formula sheets are permitted on this exam.
- Calculators are allowed and we also provide you with tables.
- All work must be shown in the answer booklet. Be sure it is complete, neat, and in order.
- Clearly indicate your final answers.
- Proper mathematical justification must be provided in order to earn full credit; correct answers with no work shown may receive no credit. Note that using the calculator does not take the place of showing your work.
- You may use standard notation; any new notation or abbreviations you use must be clearly defined.
- You have 90 minutes to complete this exam.
- On the front of your answer book, please identify your Th class.

**Problem 1** *Traveling* [**10%**] Suppose 30% of all Americans have been to Europe and 10% have been to Asia.

**a)** If travel to Europe and Asia were independent, what percentage of Americans would have traveled to at least one of these continents?

**b)** If instead 5% of Americans had been to both, what would be the conditional probability that someone who had been to Europe had also been to Asia?

**Solution:**

**a)** Since travel to Asia and Europe are are independent,

$$P(\text{Europe and Asia}) = P(\text{Europe})P(\text{Asia}) = (.3)(.1) = .03.$$

So

$$\begin{aligned} P(\text{Europe or Asia}) =& P(\text{Europe}) + P(\text{Asia}) - P(\text{Europe and Asia}) \\ =& .3 + .1 - 03 = .37. \end{aligned}$$

**b)**

$$P(\text{Asia}|\text{Europe}) = \frac{P(\text{Europe and Asia})}{P(\text{Europe}} = \frac{.05}{.30} = .167.$$

**Problem 2** *Appeals Court* [**15%**] An appeals court consists of 6 justices: Abby, Betty, Carol, Doug, Eve, and Frank. It will overturn a ruling if a majority (at least 4 justices) vote for overturn. Assume that each justice has a .4 chance of voting for an overturn on a certain case, and that the decisions of the judges are independent.

**a)** What is the probability of exactly 4 of the 6 judges voting for overturn?

**b)** What is the probability that the four women justices vote to overturn, and the other two justices do not?

**c)** Find the probability of an overturn on this case.

**Solution:**

**a)** This is $P(X = 4)$ for $X =$ Binomial(6,.4). So

$$\begin{aligned} P(X = 4) &= \binom{6}{4} (.4)^4 (.6)^2 \\ &= \binom{6}{2} (.4)^4 (.6)^2 \\ &= \frac{6 \cdot 5}{1 \cdot 2} (.0092) \\ &= 15 (.0092) \\ &= .138. \end{aligned}$$

**b)** Here we are asking about only 1 of the 15 arrangements with 4 justices voting for overturn. So the answer is $(.4)^4 (.6)^2 = .0092$.

**c)**

$$\begin{aligned} P(X \geq 4) &= P(X = 4) + P(X = 5) + P(X = 6) \\ &= .138 + \binom{6}{5} (.4)^5 (.6) + \binom{6}{6} (.4)^6 . \\ &= .138 + \binom{6}{1} (.4)^5 (.6) + \binom{6}{0} (.4)^6 . \\ &= .138 + .037 + .004 = .179. \end{aligned}$$

**Problem 3** *Automobile Manufacturing* **[20%]** A new automobile factory requires a car to be worked on by each of 125 teams as part of the final assembly process. Each team has a 40% chance of requiring exactly 1 minute to complete its task and a 60% chance of requiring exactly 2 minutes. (So it takes between 125 and 250 minutes to do the final assembly of each car.) The chance any team will need 2 minutes on a car is independent of the amount of time other teams needed on that car.

Let $X$ be a random variable describing how many more minutes than 125 that a car requires for final assembly.

**a)** Explain briefly why $X$ is a binomial(n,p) random variable. Your answer should include:

- what a Bernoulli trial is here
- values of $n$ and $p$
- a brief statement of why the quantity of interest here is what binomial(n,p) keeps track of.

**b)** Explain briefly why $X$ can be approximated by a normal distribution. Give the mean and standard deviation of this distribution.

**c)** Using normal approximation, estimate the probability that a car will require between 160 and 170 minutes (including these values) for final assembly.

**d)** What range of assembly times will the middle 80% of all cars require for final assembly?

**e)** Suppose the plant produces 100,000 cars this way during a typical year. Is it likely that at least one of these cars will require more than 195 minutes for final assembly? Briefly explain your answer.

**Solution:**

**a)** A Bernoulli trial here refers to the time for one team to complete its task. If we want X to be both binomial and count the number of extra minutes beyond 125, then a 2 minute completion time should be viewed as a "success" and a 1 minute completion time as "failure."
Now the number of extra minutes beyond 125 required is simply the total number of successes in $X = \text{Binomial}(125, .6)$. *(Binomial because it is the total number of successes ...)*
Had we tried to do the problem viewing 1-minute as a success, then $W = \text{Binomial}(125, .4)$ would count the number of successes and $Y + (125 - W) = 250 - W$ the total number of minutes needed.

**b)** We need only check the success/failure condition: $n = 125(.4) = 50 \geq 10$ and $n = 125(.6) = 75 \geq 10$. Since both of these check, we may approximate $\text{Binomial}(125, .6)$ by a normal distribution with the same mean and standard deviation. The mean here is $np = 75$ and the standard deviation is $\sqrt{125(.6)(.4)} = 5.48$.

**c)** The total number of minutes required is $Y = X + 125$. By part b), we may approximate $Y$ by an $N(75 + 125, 5.48) = N(200, 5.48)$

model. We want $P(160 \leq Y \leq 170)$. The Z score of 170 is

$$\frac{170 - 200}{5.48} = -5.47$$

and the Z score of 160 is

$$\frac{160 - 200}{5.48} = -7.30.$$

Each of these Z scores is off the charts in table Z, so both $P(Z < -5.47)$ and $P(Z < -7.30)$ are less than .0001.
Hence $P(160 \leq Y \leq 170) < .0001$; the probability is essentially 0.
**Remark:** Sorry for the distractingly extreme Z scores! In making up this problem, I had desired a mean of 175 for $X + 125$, but interchanged the .4 and .6 probabilities required to achieve that. Unfortunately, we did not catch this oversight during proof-reading.

**d)** Using an $N(0, 1)$ picture, we see that the middle 80% corresponds to $-1.28 \leq Z \leq 1.28$. Hence the corresponding range for $Y$ is from $200 - 1.28(5.48) = 200 - 7.01 = 192.99$ to $200 + 1.28(5.48) = 200 + 7.01 = 207.01$ minutes.

**e)** Since the mean is 200 minutes, we expect that more than half the cars will require more than 195 minutes.
**Remark:** Had the mean been 175 as I thought when making up the question, this would have been a closer call. The Z score would have been $\frac{20}{5.48} = 3.65$ with $P(Z > 3.65) = 10^{-4}$. But with $10^5$ cars, we'd still have expected to see something like 10 cars needing this much time.

**Problem 4** *Random Variables* [**20%**] Let $X$ be a random variable with probability model

| **X** | **prob** |
|---|---|
| 0 | .4 |
| 2 | .6 |

and $Y$ a random variable with probability model

| **Y** | **prob** |
|---|---|
| 0 | .4 |
| 3 | .2 |
| 6 | .4 |

Assume $X$ and $Y$ are independent random variables.
**a)** What are the possible values for the random variable $X + Y$?

**b)** What is the probability that the random variable $X + Y$ takes on the value 5? (i.e. $P(X + Y = 5)$.)

**c)** Find the mean and standard deviation of the random variable $X$.

**d)** Let $X_1$ and $X_2$ be two independent copies of the random variable $X$. Find the standard deviation of the random variable $2X_1 + X_2$.

**Solution:**

**a)** The possible values of $X + Y$ are 0, 3, 6, 2, 5, and 8. These numbers are obtained by adding either 0 or 2 (the possible values of $X$) to the three possible values of $Y$.

**b)** The only way $X+Y$ can come out to 5 is if $X = 2$ and $Y = 3$. By independence, $P(X + Y = 5) = P(X = 2)P(Y = 3) = .6 \cdot .2 = .12$.

**c)** A direct calculation is not hard. One can also observe that $X = 2B$ where $B$ is a Bernoulli random variable with parameter .6. ($B$ is the same as Binomial$(1, .6)$.) Our basic formulas tell us the mean of $B$ is .6 and the standard deviation is $\sqrt{.6 \cdot .4} = \sqrt{.24} = .49$. So the mean of $X$ is $2(.6) = 1.2$ and the standard deviation is $2(.49) = .98$.

**d)** By independence

$$\begin{aligned}
\mathrm{Var}(2X_1 + X_2) =& \mathrm{Var}(2X_1) + \mathrm{Var}(X_2) \\
=& 2^2\mathrm{Var}(X_1) + \mathrm{Var}(X_2) \\
=& 5\mathrm{Var}(X) \\
=& 5(.98)^2 \\
=& 4.802.
\end{aligned}$$

So the standard deviation of $2X_1 + X_2$ is $\sqrt{4.802} = 2.19$.

**Problem 5** *Auto Insurance* [**15%**] An auto insurance policy specifies that the company will pay \$1000 for a minor accident and \$5000 for a major accident. For 10% of all policy holders, the first (and possibly only) accident they have during the year is a minor one. For 5% of all policy holders, the first (and possibly only) accident they have during the year is a major one. (The rest of the policy holders have no accidents all year.) For those who have that first accident during the year, there is a 40% chance of another accident. There is an equal chance that the second accident anyone has will be major or minor. After the second accident, payment is made for that accident and the policy is canceled.

**a)** Give a probability model for the random variable describing the amount of money the company will have to pay out for one policy holder.

**b)** Find the mean of your random variable in part a).

**c)** If the company charges $750 per year for such a policy, would you expect the company to make a profit? (Neglect administrative costs.) Briefly explain.

**Solution:**
**a)** There are several possibilities:

- (a) Exactly one minor accident - pay \$1000.
- (b) Exactly one major accident - pay \$5000.
- (c) First accident minor followed by a minor - pay \$2000.
- (d) First accident minor followed by a major - pay \$6000.
- (e) First accident major followed by a minor - pay \$6000.
- (f) First accident major followed by a major - pay \$10000.
- (g) No accidents - pay \$0.

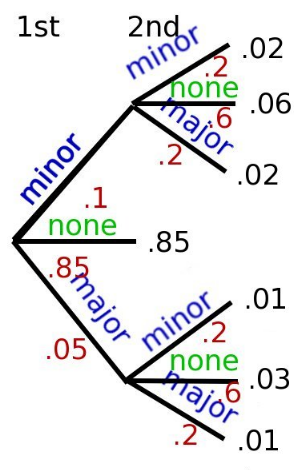
To determine the probabilites of these, we need to use the general multiplication rule

$$P(A \text{ and } B) = P(A)P(B|A)$$

together with the specific info

- P(no second accident | first accident) = 60%
- P(minor second accident | first accident) = 20%
- P(major second accident | first accident) = 20%

Like sample exam question #5 from Spring '08, this could, if desired for clarity, be put into a probability tree. But one need not do this. Here is the tree one might draw:



Consequently, the probability model for the payout $X$ is:

| **X** | **prob** |
|---|---|
| 0 | .85 |
| 1K | .06 |
| 2K | .02 |
| 5K | .03 |
| 6K | .03 |
| 10K | .01 |

**b)** The expectation of $X$ is

$$
\begin{aligned}
.06(1K) + .02(2K) + .03(5K) + .03(6K) + .01(10K) &= \\
60 + 40 + 150 + 180 + 100 &= \\
&= \$530.
\end{aligned}
$$

**c)** The RV $750 - X$ describes the gross profit of the company. Since it has a mean of $\$750 - \$530 = \$220 > 0$, we expect the company to make a profit.

**Problem 6** *Swimming in Polluted Waters* [**20%**] In 1989, the New South Wales Department of Health studied whether swimming in polluted waters might be leading to increased acute infectious illness. As part of this study, visitors to two beaches were polled later in the season to see if they had suffered an illness within some time after their visit. Some visitors had swum in polluted water; some had swum only in non-polluted water; some had only sunbathed.

Of those who had stayed well:

- 5% had swum in polluted water.
- 60% had swum only in non-polluted water.
- 35% had just sunbathed.

Of those who had gotten sick:

- 8% had swum in polluted water.
- 70% had swum only in non-polluted water.
- 22% had just sunbathed.

Overall, about 24% of all people polled had gotten sick.

**a)** Construct a tree diagram for this situation. Be sure and explain any abbreviations or non-standard notation you use.

**b)** For the group studied, what was the probability someone had swum in polluted waters? How about the probability someone had swum only in non-polluted waters?

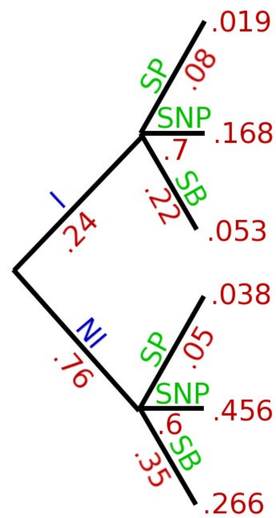
**c)** What is the conditional probability of someone getting sick given that they had swum in polluted waters?

**Solution:**

**a)** We use the following notation:

- SP means swum in polluted water.
- SNP means had swum only in non-polluted water.
- SB means just sunbathed.
- I means got ill.
- NI means did not get ill.

Below is the tree diagram in which the first stage refers to whether or not someone had gooten ill, and the second stage which kind of beach experience they'd had.




**b)** From the tree diagram

$$P(SP) = .019 + .038 = .057$$
$$P(SNP) = .168 + .456 = .624$$

**c)**

$$P(I|SP) = \frac{P(I \text{ and } SP)}{P(SP)} = \frac{.019}{.057} = .333.$$