Math 1220 **Prelim 2 solutions** **October 29, 2009**
Fall 2009

1. (15 points) Find the area of the surface generated by revolving about the y-axis the curve

$$y = \tfrac{1}{2}(x^2 + 1),\ 0 \le x \le 1.$$

We will use formula $S = \int_c^d 2\pi x\sqrt{1 + (dx/dy)^2}\,dy$.
Re-write the curve as $x^2 = 2y - 1$, so if $x = 0$, then $y = \tfrac{1}{2}$, and if $x = 1$, then $y = 1$.
$2y - 1 \ge 0$ for $\tfrac{1}{2} \le y \le 1$ and for these values $x = \sqrt{2y-1}$.

$$\frac{dx}{dy} = \frac{1}{\sqrt{2y-1}} \quad \text{and} \quad \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \frac{1}{2y-1}} = \frac{\sqrt{2y}}{\sqrt{2y-1}}$$

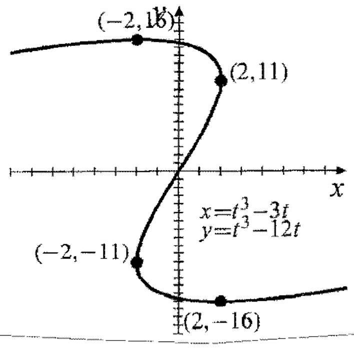
$$S = 2\pi \int_{1/2}^{1} \sqrt{2y-1} \cdot \frac{\sqrt{2y}}{\sqrt{2y-1}}\,dy = 2\sqrt{2}\pi \left.\frac{y^{3/2}}{3/2}\right|_{1/2}^{1} = \frac{2\pi}{3}(2\sqrt{2} - 1)$$

2. (20 points) Determine the points where the given parametric curve has horizontal or vertical tangent lines, and sketch the curve.

$$x = t^3 - 3t,\ y = t^3 - 12t$$

$\dfrac{dx}{dt} = 3(t^2 - 1) \quad \dfrac{dy}{dt} = 3(t^2 - 4)$
Horizontal tangent at $t = \pm 2$, i.e., at $(2, -16)$ and $(-2, 16)$.
Vertical tangent at $t = \pm 1$, i.e., at $(2, 11)$ and $(-2, -11)$.

Slope $\dfrac{dy}{dx} = \dfrac{t^2 - 4}{t^2 - 1} \begin{cases} > 0 & \text{if } |t| > 2 \text{ or } |t| < 1 \\ < 0 & \text{if } 1 < |t| < 2 \end{cases}$
Slope $\to 1$ as $t \to \pm\infty$.



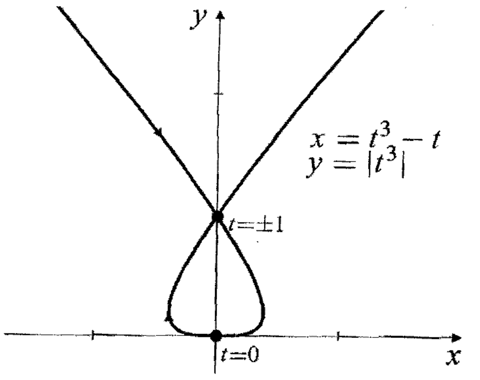
3. (20 points) Find the volume of the solid generated by rotating about the y-axis the closed loop of the curve:

$$x = t^3 - t,\ y = |t^3|.$$

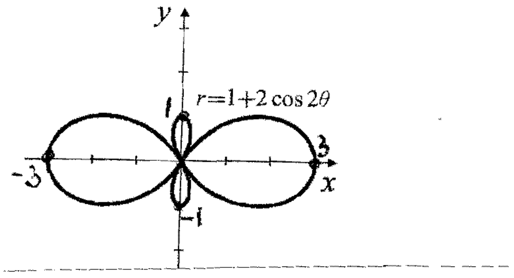
The curve $x = t^3 - t$, $y = |t^3|$ is symmetric about $x = 0$ since x is an odd function and y is an even function. Its self-intersection occurs at a nonzero value of t that makes $x = 0$, namely, $t = \pm 1$. ~~The area of the loop is~~

The volume of revolution about the y-axis is

$$\begin{aligned} V &= \pi \int_{t=0}^{t=1} x^2\,dy \\ &= \pi \int_0^1 (t^6 - 2t^4 + t^2)3t^2\,dt \\ &= 3\pi \int_0^1 (t^8 - 2t^6 + t^4)\,dt \\ &= 3\pi\left(\frac{1}{9} - \frac{2}{7} + \frac{1}{5}\right) = \frac{8\pi}{105} \text{ cu. units.} \end{aligned}$$



4. (15 points) a) Sketch the polar graph of the equation, $r = 1 + 2\cos(2\theta)$.



(15 points) b) Find the area enclosed by this curve.

Area of a small loop:

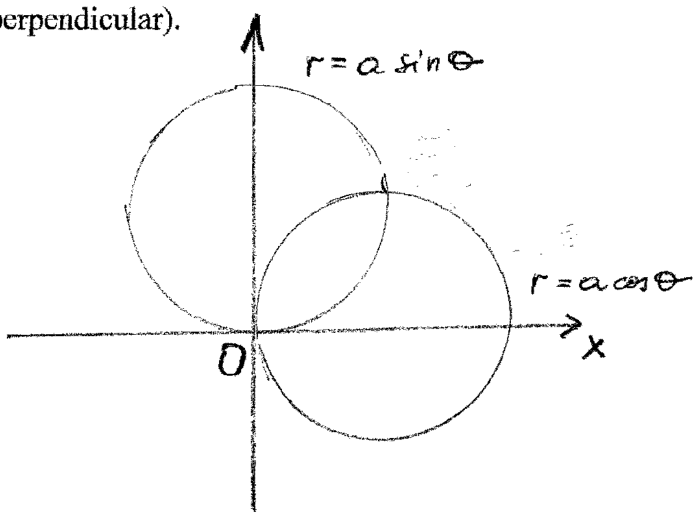
$$\begin{aligned} A_1 &= 2 \times \frac{1}{2}\int_{\pi/3}^{\pi/2} (1 + 2\cos(2\theta))^2\,d\theta \\ &= \int_{\pi/3}^{\pi/2} [1 + 4\cos(2\theta) + 2(1 + \cos(4\theta))]\,d\theta \\ &= \left(3\theta + 2\sin(2\theta) + \frac{1}{2}\sin(4\theta)\right)\Big|_{\pi/3}^{\pi/2} \\ &= \frac{\pi}{2} - \frac{3\sqrt{3}}{4} \text{ sq. units.} \end{aligned}$$

Area of a large loop:

$$\begin{aligned} A_2 &= 2 \times \frac{1}{2}\int_0^{\pi/3} (1 + 2\cos(2\theta))^2\,d\theta \\ &= \int_0^{\pi/3} [1 + 4\cos(2\theta) + 2(1 + \cos(4\theta))]\,d\theta \\ &= \left(3\theta + 2\sin(2\theta) + \frac{1}{2}\sin(4\theta)\right)\Big|_0^{\pi/3} \\ &= \pi + \frac{3\sqrt{3}}{4} \text{ sq. units.} \end{aligned}$$

Total area $= 2(A_1 + A_2) = 2\left(\frac{\pi}{2} - \frac{3\sqrt{3}}{4} + \pi + \frac{3\sqrt{3}}{4}\right) = \underline{\underline{3\pi}}$ sq. units

5. (15 points) Show that the curves $r = a\sin\theta$ and $r = a\cos\theta$ intersect at right angles (show that the tangent lines at the point(s) of intersection of these curves are perpendicular).



Curves intersect at 2 points, but because of symmetry, we can look only at one of those points:

Points of intersection:

$a\sin\theta = a\cos\theta$ or $(0;0)$

or when $\tan\theta = 1$, $(\cos\theta \neq 0)$

$\theta = \frac{\pi}{4}$.

Let us consider $\theta = \frac{\pi}{4}$:

$$K_1 = \tan\psi_1 = \left.\frac{a\sin\theta}{a\cos\theta}\right|_{\theta=\frac{\pi}{4}} = \tan\left(\frac{\pi}{4}\right) = 1$$

$$K_2 = \tan\psi_2 = \left.\frac{a\cos\theta}{-a\sin\theta}\right|_{\theta=\frac{\pi}{4}} = -\cot\left(\frac{\pi}{4}\right) = -1$$

$K_1 \cdot K_2 = -1 \Rightarrow$ tangent lines are perpendicular and therefore the curves intersect at right angles.