Math 1220 **Final exam solutions** **December 16, 2009**
Fall 2009

If you found a different solution and did everything correctly, you will receive a full credit.

1. (15 pts) Evaluate following integrals:

a) $\underline{\int \cos(\ln x)dx} =$

Let $u = \cos(\ln x)$ and $dv = dx$, then $du = \sin(\ln x)(dx/x)$ and $v = x$ and integrating by parts:

$= x\cos(\ln x) + \int x \sin(\ln x)\frac{dx}{x}$

Integrating by parts once more: let $u = \sin(\ln x)$ and $dv = dx$, then $du = \cos(\ln x)\,dx$ and $v = x$

$\underline{= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\,dx}$

Now from underlined follows:

$\int \cos(\ln x)dx = \frac{1}{2}x[\sin(\ln x) + \cos(\ln x)] + C.$

b) $\int_{-1}^{2} x^{-4}dx = \int_{-1}^{0} x^{-4}\,dx + \int_{0}^{2} x^{-4}\,dx$. Let us look at the first integral

$\int_{-1}^{0} x^{-4}dx = \lim_{a\to 0^-} \int_{-1}^{a} x^{-4}dx = \lim_{a\to 0^-} -1/3\, x^{-3}\Big|_{-1}^{a} = \infty,$

so the original integral diverges.

c) $\int_0^\infty \frac{e^x}{e^{2x}+1}dx$ substitution $u = e^x$, then $du = e^x dx$, and if $x=0$, then $u=1$, and if $x=\infty$, then $u = \infty$

$= \int_1^\infty \frac{du}{u^2+1} = \lim_{b\to\infty} \tan^{-1} u \Big|_1^b = \pi/2 - \pi/4 = \pi/4.$

2. (10 pts) Find $f(\pi/2)$, if f is a positive and continuous function and the area under the curve $y = f(x)$, from $x = 0$ to $x = a$ is $\frac{1}{2}a^2 + \frac{1}{2}a\sin a + \frac{1}{2}\pi\cos a$.

We know that the area can be expressed as $\int_0^a f(x)\,dx = \frac{1}{2}a^2 + \frac{1}{2}a\sin a + \frac{1}{2}\pi\cos a$.
The Fundamental Theorem of Calculus tells us

$f(a) = \frac{d}{da}\left(\int_0^a f(x)\,dx\right) = \frac{d}{da}\left(\frac{1}{2}a^2 + \frac{1}{2}a\sin a + \frac{1}{2}\pi\cos a\right) =$

$= a + \frac{1}{2}a\cos a + \frac{1}{2}\sin a - \pi/2\sin a$, so if $a = \pi/2$, then $f(\pi/2) = \frac{1}{2}$.

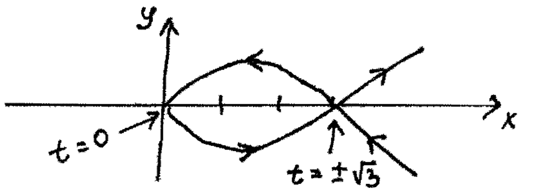
3. (10 pts) Find the equation of the curve that passes through the point (0,3) and has a slope $\frac{x}{1+y^2}$ at any point (x, y) on it.

Remember that slope k = dy/dx, we can write initial value problem dy/dx = $\frac{x}{1+y^2}$, $y(0)=3$. Separating variables: $(1+y^2)\,dy = x\,dx$ and integrating both sides $1/3\, y^3 + y = ½\, x^2 + C$,

From $y(0)=3 \Rightarrow 1/3\,(3)^3 + 3 = C$ or $C = 12$, thus the equation of the curve is

$1/3\, y^3 + y = ½\, x^2 + 12$.

4. (10 pts) Sketch the graph of the curve $x = t^2$, $y = t^3/3 - t$ and calculate the length of the part of the curve that is the enclosed loop.



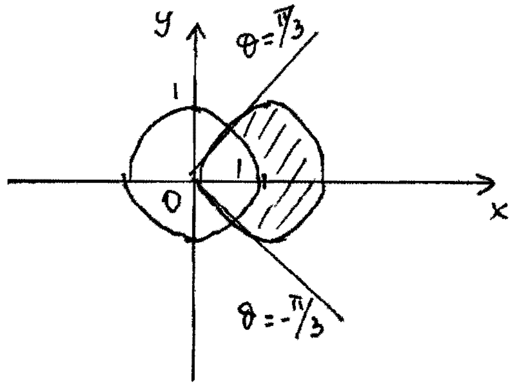
Length (by symmetry) $L = 2\int_0^{\sqrt{3}} \sqrt{(2t)^2+(t^2-1)^2}\,dt =$

$$= 2\int_0^{\sqrt{3}} \sqrt{t^4-2t^2+4t^2+1}\,dt = 2\int_0^{\sqrt{3}} \sqrt{(t^2+1)^2}\,dt =$$

$$= 2\int_0^{\sqrt{3}} (t^2+1)\,dt = 2\left(\frac{1}{3}t^3+t\right)\Big|_0^{\sqrt{3}} = 2\left(\frac{3\sqrt{3}}{3}+\sqrt{3}\right) = 4\sqrt{3} \text{ units}$$

5. (10 pts) Find the area of the region inside the curve $r = 2\cos\theta$ but outside the curve $r = 1$.

$r = 2a\cos\theta$ is the polar equation of the circle of radius a with center $(a,0)$.
Thus $r = 2\cos\theta$ is a circle of radius 1 with center (1,0). $r = 1$ is the circle with radius 1 centered at the origin. We want to compute shaded region:



The two circles intersect at the points where $r = 2\cos\theta = 1$ or $\cos\theta = \frac{1}{2}$, $\theta = \pm \pi/3$.

The area of the shaded region is equal to

$$A = \tfrac{1}{2}\int_{-\pi/3}^{\pi/3} (2\cos\theta)^2\, d\theta - \tfrac{1}{2}\int_{-\pi/3}^{\pi/3} (1)^2\, d\theta = \int_0^{\pi/3} (4\cos^2\theta - 1)\, d\theta \qquad (2\cos^2\theta = 1+\cos 2\theta)$$

$$= \int_0^{\pi/3} (2\cos 2\theta + 1)\, d\theta = (\sin 2\theta + \theta)\Big|_0^{\pi/3} = \frac{\sqrt{3}}{2} + \frac{\pi}{3} \quad \text{sq. units}$$

6. (10 pts) Find the volume of the solid generated by rotating around the x-axis the region in the first quadrant enclosed by the curve $y = \dfrac{5}{x\sqrt{5-x}}$ and the lines $x = 1$ and $x = 4$.

$$\text{Volume} = \pi \int_1^4 \left(\frac{5}{x\sqrt{5-x}}\right)^2 dx = \pi \int_1^4 \frac{25}{x^2(5-x)}\, dx$$

Partial fractions: $\dfrac{25}{x^2(5-x)} = \dfrac{Ax+B}{x^2} + \dfrac{C}{5-x} = \dfrac{(C-A)x^2 + (5A-B)x + 5B}{x^2(5-x)}$

Thus $C - A = 0 \Rightarrow C = A$ $\quad C = 1$

$5A - B = 0 \Rightarrow B = 5A$ $\quad A = 1$

$5B = 25 \Rightarrow B = 5$

$$\frac{Ax+B}{x^2} = \frac{x+5}{x^2} = \frac{1}{x} + \frac{5}{x^2}$$

Volume $= \pi \int_1^4 \left(\frac{1}{x} + \frac{5}{x^2} + \frac{1}{5-x}\right) dx = \pi\left(\ln|x| - \frac{5}{x} - \ln|5-x|\right)\Big|_1^4 =$

$= \pi\left(4\ln 2 + \frac{15}{4}\right)$ cu. units

7. (10 pts) The graph of $y = x^{-1/3}$, $0<x\le 1$, is rotated about the x-axis. Show that the area of the resulting surface of revolution is infinite.

If we let $f(x) = x^{-1/3}$ then $f'(x) = -\frac{1}{3}x^{-4/3}$

And the surface area is given by

$$S = \int_0^1 2\pi x^{-1/3}\left(1 + \left(-\frac{x^{-4/3}}{3}\right)^2\right)^{1/2} dx = \frac{2\pi}{3}\lim_{a\to 0^+} \int_a^1 x^{-1/3}\sqrt{9 + x^{-8/3}}\, dx$$

Notice that $x^{-1/3}(9 + x^{-8/3})^{1/2} > x^{-1/3}(x^{-8/3})^{1/2} = x^{-5/3}$.

We conclude that $S \geqslant \frac{2\pi}{3} \lim_{R \to 0^+} \int_R^1 x^{-5/3}\,dx = \frac{2\pi}{3} \lim_{R\to 0^+} \left(\frac{x^{-2/3}}{-2/3} \right)\Big|_R^1 =$

$= \pi \lim_{R \to 0^+} \left(R^{-2/3} - 1 \right) = \infty .$

Therefore, the area of the surface of revolution is infinite.

8. (10 pts) Find the radius and interval of convergence, and identify the values of x where the series $\sum_{n=1}^{\infty} \frac{(x+4)^n}{n\,3^n}$ converges absolutely and conditionally.

Center is -4, radius $R = \lim_{n \to \infty} \left| \frac{\frac{1}{n \cdot 3^n}}{\frac{1}{(n+1)\cdot 3^{n+1}}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\cdot 3^{n+1}}{n \cdot 3^n} \right| = \lim_{n \to \infty} \left(3 \cdot \frac{n+1}{n} \right) = 3.$

Interval of convergence $-3 < x+4 < 3$ or $-7 < x < -1$.

At $x = -7$, series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ conditionally converges

At $x = -1$, series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

9. (5 pts) Use power series to evaluate $\lim_{x\to 0} \frac{7 \sin x}{e^{2x}-1} = \lim_{x \to 0} \frac{7x - \frac{7x^3}{3!} + \frac{7x^5}{5!} - \ldots}{-1+1+2x+\frac{4x^2}{2}+\frac{8x^3}{6}+\ldots} =$

$= \lim_{x\to 0} \frac{x\left(7 - \frac{7x^2}{6} + \ldots\right)}{x\left(2 + 2x + \ldots\right)} = \frac{7}{2}.$

10. (10 pts) Show that the series $\sum_{n=1}^{\infty} \left(\sin\frac{1}{2n} - \sin\frac{1}{2n+1}\right)$ converges.

Notice that we can rewrite series as an alternating series $\sum_{n=2}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right).$

First, we know the series does not converge absolutely using the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$ (which diverges) $\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{k \to 0} \frac{\sin k}{k} = 1 < \infty$

However, the series conditionally converges by the alternating series test:

(1) the terms are alternating;

(2) $0 < \frac{1}{n} \le 1$ for all $n \geqslant 1$, and $\sin x$ is increasing on $[0,1]$ so $\sin\left(\frac{1}{n+1}\right) < \sin\frac{1}{n}$

(3) $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0.$