Math 1220 **Prelim II Solutions** **Fall 2008**

1. The surface area is

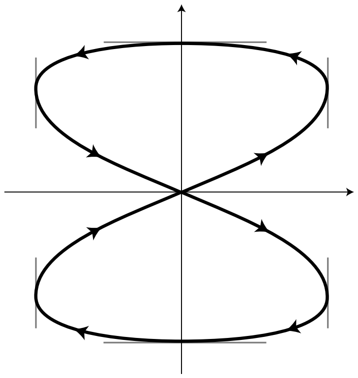
$$2\pi\int_0^1 x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_0^1 x\sqrt{1+\frac{1}{x^2}}\,dx = 2\pi\int_0^1 \sqrt{x^2+1}\,dx$$

We substitute $x\tan\theta$, $dx=\sec^2\theta\,d\theta$, to convert the integral into

$$\begin{aligned}2\pi\int\sec^3\theta\,d\theta &= \pi\big[\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\big]\\ &= \pi\big[x\sqrt{x^2+1}+\ln(x+\sqrt{x^2+1}\big]\Big|_0^1 = \pi\big[\sqrt{2}+\ln(1+\sqrt{2})\big]\end{aligned}$$

2. The curve $x=\sin 2t$, $y=\sin t$ has slope

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\cos t}{2\cos 2t}$$



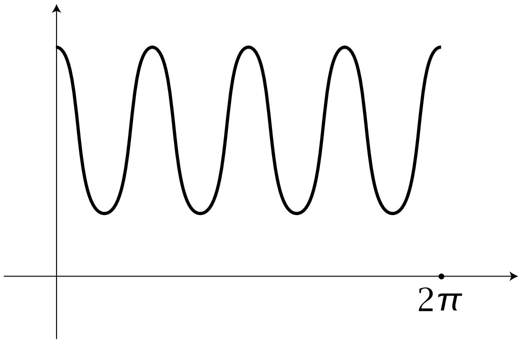
(a) The curve passes through the origin when $\sin 2t=0$ and $\sin t=0$, so $t=0,\pi$, and 2π. The slope at $t=0$ and 2π is $1/2$ and the slope at $t=\pi$ is $-1/2$.

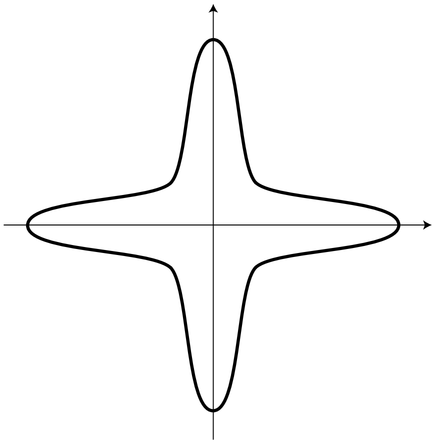
(b) The horizontal tangents occur when the slope is 0, so $dy/dt=0$, i.e., $\cos t=0$ so $t=\pi/2$ and $3\pi/2$. These are the points $(x,y)=(0,1)$ and $(0,-1)$. The vertical tangents occur when the slope is ∞ so $dx/dt=0$, i.e., $\cos 2t=0$ so $t=\pi/4,3\pi/4,5\pi/4,7\pi/4$. These are the points $(x,y)=(1,1/\sqrt{2}),(-1,1/\sqrt{2}),(1,-1/\sqrt{2}),(-1,-1/\sqrt{2})$.

(c) A sketch of the curve is shown above. It starts at the origin when $t=0$, then moves northeast to the point with a vertical tangent, then northwest to the point with a horizontal tangent, then southwest to the vertical tangent, then southeast back to the origin, then it continues symmetrically through the third and fourth quadrants back to the origin again.

(d) The arclength is $\displaystyle\int_0^{2\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_0^{2\pi}\sqrt{4\cos^2 2t+\cos^2 t}\,dt$

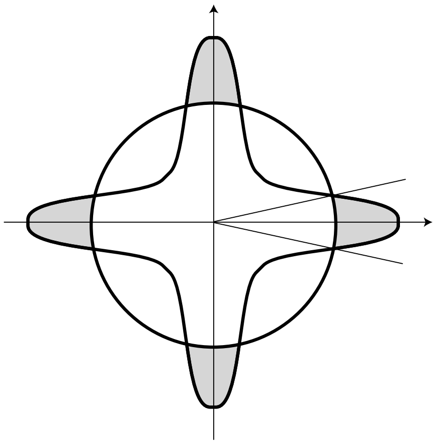
3. (a) To sketch the polar curve $r = 2 + \cos 4\theta$ we can first graph the function $2 + \cos 4\theta$ in rectangular coordinates, as in the first figure below. Then the graph in polar coordinates looks like the second figure.





(b) The area inside the curve is

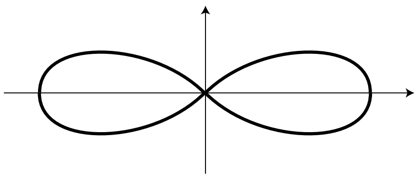
$$\frac{1}{2}\int_0^{2\pi} r^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(4 + 4\cos 4\theta + \cos^2 4\theta)\,d\theta = \frac{1}{2}\int_0^{2\pi}\Big(4 + 4\cos 4\theta + \frac{1}{2} + \frac{\cos 8\theta}{2}\Big)\,d\theta$$
$$= \frac{1}{2}\Big(4\theta + \sin 4\theta + \frac{\theta}{2} + \frac{\sin 8\theta}{16}\Big)\Big|_0^{2\pi} = \frac{9\pi}{2}$$



(c) The part of the region outside the circle $r = 2$ consists of 4 separate pieces. By symmetry we can compute the area of half of one of the pieces and then multiply by 8. The curve $r = 2 + \cos 4\theta$ intersects the circle $r = 2$ where $2 + \cos 4\theta = 2$, so $\cos 4\theta = 0$, hence the first place where the curve intersects the circle is at $\theta = \pi/8$. Using the computation in part (b), the area inside the curve $r = 2 + \cos 4\theta$ for $0 \le \theta \le \pi/8$ is

$$\frac{1}{2}\Big(4\theta + \sin 4\theta + \frac{\theta}{2} + \frac{\sin 8\theta}{16}\Big)\Big|_0^{\pi/8} = \frac{1}{2}\Big(\frac{\pi}{2} + 1 + \frac{\pi}{16} + 0\Big) = \frac{\pi}{4} + \frac{1}{2} + \frac{\pi}{32}$$

We have to subtract from this the area of the sector of the circle with $0 \le \theta \le \pi/8$. This sector is one sixteenth of a circle of radius 2 so the area of this sector is $\pi/4$. Subtracting this from the area calculated above gives $\frac{1}{2} + \frac{\pi}{32}$. Multiplying by 8 yields the final answer of $4 + \frac{\pi}{4}$.



4. The lemniscate $r^2 = \cos 2\theta$ is shown in the figure. By symmetry we can compute the surface area when the right half is rotated about the x-axis and then double this. We get the right half of the rotated surface by rotating the part of the lemniscate with $0 \le \theta \le \pi/4$. Thus the total area is

$$2 \cdot 2\pi \int_0^{\pi/4} y\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = 4\pi \int_0^{\pi/4} r\sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

We have $r = (\cos 2\theta)^{1/2}$ so $dr/d\theta = (\cos 2\theta)^{-1/2}\sin 2\theta$ and the integral becomes

$$\begin{aligned} 4\pi \int_0^{\pi/4} \sqrt{\cos 2\theta}\,\sin\theta \sqrt{\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}}\, d\theta = 4\pi \int_0^{\pi/4} \sin\theta\, d\theta &= 4\pi(-\cos\theta)\Big|_0^{\pi/4} \\ &= 4\pi\Big(-\frac{1}{\sqrt{2}} + 1\Big) \end{aligned}$$

5. We are given that there are $2n$ petals, so each petal is contained in a sector of the circle of angle $2\pi/2n = \pi/n$. The area of one petal is

$$\frac{1}{2}\int_0^{\pi/n} \sin^2 n\theta\, d\theta = \frac{1}{4}\int_0^{\pi/n} \big(1 - \cos 2n\theta\big)\, d\theta = \frac{1}{4}\Big(\theta - \frac{\sin 2n\theta}{2n}\Big)\Big|_0^{\pi/n} = \frac{\pi}{4n}$$

There are $2n$ petals, so their total area is $2n\Big(\dfrac{\pi}{4n}\Big) = \dfrac{\pi}{2}$. The area of the circumscribed circle is π, so the petals cover exactly one-half of the area of the circle. (It is interesting that the answer does not depend on n.)