1. Compute the following integrals:

(a) $\int \tan^3 x \sec^3 x \, dx$. Using the identity $\tan^2 x + 1 = \sec^2 x$ this becomes

$$\begin{aligned}\int \tan x(\sec^2 x - 1)\sec^3 x \, dx &= \int \sec^4 x(\sec x \tan x)\, dx - \int \sec^2 x(\sec x \tan x)\, dx \\ &= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C\end{aligned}$$

(b) $\int \frac{\sin^3 x}{\cos^2 x}\, dx$. Using $\sin^2 x + \cos^2 x = 1$ this becomes

$$\int \frac{\sin x(1-\cos^2 x)}{\cos^2 x}\, dx = \int \frac{\sin x}{\cos^2 x}\, dx - \int \sin x \, dx = \frac{1}{\cos x} + \cos x + C$$

(c) $\int \frac{x^3 + x + 1}{\sqrt{x^2-4}}\, dx$. Let $x = 2\sec\theta$, so $\sqrt{x^2-4} = 2\tan\theta$ and $dx = 2\sec\theta\tan\theta$. Then the integral becomes

$$\begin{aligned}\int \frac{(8\sec^3\theta + 2\sec\theta + 1)(2\sec\theta\tan\theta)}{2\tan\theta}\, d\theta &= \int (8\sec^4\theta + 2\sec^2\theta + \sec\theta)\, d\theta \\ &= \int (8\sec^2\theta(\tan^2\theta + 1) + 2\sec^2\theta + \sec\theta)\, d\theta \\ &= \int (8\sec^2\tan^2\theta + 10\sec^2\theta + \sec\theta)\, d\theta \\ &= \frac{8\tan^3\theta}{3} + 10\tan\theta + \ln|\sec\theta + \tan\theta| + C \\ &= \frac{(x^2-4)^{3/2}}{3} + 5(x^2-4)^{1/2} + \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C\end{aligned}$$

(d) $\int \frac{x^{1/3}}{(x^{2/3}+1)^2}\, dx$. Let $u = x^{1/3}$, so $u^3 = x$ and $dx = 3u^2\, du$. The integral then becomes $3\int \frac{u^3}{(u^2+1)^2}\, du$. Now we use the method of partial fractions:

$$\begin{aligned}\frac{u^3}{(u^2+1)^2} &= \frac{Au+B}{u^2+1} + \frac{Cu+C}{(u^2+1)^2} \\ &= \frac{(Au+B)(u^2+1) + Cu + D}{(u^2+1)^2} \\ &= \frac{Au^3 + Bu^2 + (A+C)u + (B+D)}{(u^2+1)^2}\end{aligned}$$

Equating corresponding coefficients gives $A = 1$, $B = 0$, $A + C = 0$ hence $C = -1$, and $B + D = 0$ hence $D = 0$. The integral is thus equal to

$$
\begin{aligned}
3\int \frac{u\,du}{u^2+1} - 3\int \frac{u\,du}{(u^2+1)^2} &= \frac{3}{2}\ln(u^2+1) + \frac{3}{2}(u^2+1)^{-1} + C \\
&= \frac{3}{2}\ln(x^{2/3}+1) + \frac{3}{2}(x^{2/3}+1)^{-1} + C
\end{aligned}
$$

(e) $\displaystyle\int \frac{x^4+2x^3+2x^2+3}{x^4-1}\,dx$. Before using partial fractions we do long division to get

$$\frac{x^4+2x^3+2x^2+3}{x^4-1} = 1 + \frac{2x^3+2x^2+4}{x^4-1}$$

The denominator factors as $(x-1)(x+1)(x^2+1)$ so the partial fractions decomposition is

$$
\begin{aligned}
\frac{2x^3+2x^2+4}{x^4-1} &= \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} \\
&= \frac{A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)}{x^4-1} \\
&= \frac{(A+B+C)x^3 + (A-B+D)x^2 + (A+B-C)x + (A-B-D)}{x^4-1)}
\end{aligned}
$$

This leads to the equations

$$A + B + C = 2 \qquad (1)$$

$$A - B + D = 2 \qquad (2)$$

$$A + B - C = 0 \qquad (3)$$

$$A - B - D = 4 \qquad (4)$$

Adding equations (1) and (3) gives $2A + 2B = 2$ so $A + B = 1$. Adding equations (2) and (4) gives $2A - 2B = 6$ so $A - B = 3$. Combining $A + B = 1$ and $A - B = 3$ yields $A = 2$, $B = -1$. Substituting back in equations (1) and (2) then gives $C = 1$ and $D = -1$. Thus the original integral equals

$$
\begin{aligned}
&\int dx + 2\int \frac{dx}{x-1} - \int \frac{dx}{x+1} + \int \frac{x\,dx}{x^2+1} - \int \frac{dx}{x^2+1} \\
&\qquad = x + 2\ln|x-1| - \ln|x+1| + \frac{1}{2}\ln|x^2+1| - \tan^{-1}x + C
\end{aligned}
$$

(f) $\int x\left(\tan^{-1} x\right)^2 dx$. This one is a bit harder than the previous integrals. The most likely thing to try is integration by parts. To split $x\left(\tan^{-1} x\right)^2 dx$ into the form $u\,dv$ we can follow the strategy of making dv include the largest part that we can easily integrate, so we try $dv = x\,dx$, so $v = x^2/2$ and $u = (\tan^{-1} x)^2$, hence $du = \dfrac{2\tan^{-1} x}{1+x^2}\,dx$. Now the integral becomes

$$\int x\left(\tan^{-1} x\right)^2 dx = \frac{x^2}{2}\left(\tan^{-1} x\right)^2 - \int \frac{x^2}{1+x^2}\tan^{-1} x\,dx$$

We can simplify $\dfrac{x^2}{1+x^2}$ using long division to get $\dfrac{x^2}{1+x^2} = 1 - \dfrac{1}{1+x^2}$. This gives us

$$\int x\left(\tan^{-1} x\right)^2 dx = \frac{x^2}{2}\left(\tan^{-1} x\right)^2 - \int \tan^{-1} x\,dx + \int \frac{\tan^{-1} x}{1+x^2}\,dx$$

This last integral can be computed by a simple substitution $u = \tan^{-1} x$, yielding

$$\int \frac{\tan^{-1} x}{1+x^2}\,dx = \frac{1}{2}\left(\tan^{-1} x\right)^2$$

For the other integral $\int \tan^{-1} x\,dx$ we use integration by parts again: $u = \tan^{-1} x$, $du = \dfrac{1}{1+x^2}\,dx$, $dv = dx$, $v = x$, so

$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2)$$

Putting everything together, we get

$$\int x\left(\tan^{-1} x\right)^2 dx = \frac{x^2}{2}\left(\tan^{-1} x\right)^2 - x\tan^{-1} x + \frac{1}{2}\ln(1+x^2) + \frac{1}{2}\left(\tan^{-1} x\right)^2 + C$$

This could be simplified slightly by combining the first and fourth terms.

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2. Determine whether the following improper integrals converge or diverge:

(a) $\displaystyle\int_e^\infty \frac{dx}{x\ln x}$. By a substitution $u = \ln x$ this becomes $\ln(\ln x)\,\Big|_e^\infty = \infty$ so the integral diverges.

(b) $\displaystyle\int_e^\infty \frac{dx}{(\ln x)^2}$. We can compare this with the integral in part (a). We have $\ln x < x$ when $x \geq e$ (look at the graphs of $y = \ln x$ and $y = x$), hence $\displaystyle\int_e^\infty \frac{dx}{(\ln x)^2} > \int_e^\infty \frac{dx}{x\ln x}$. The latter integral diverges by part (a), so $\displaystyle\int_e^\infty \frac{dx}{(\ln x)^2}$ also diverges.

(c) $\int_1^e \frac{dx}{\ln x}$. This is improper at $x = 1$. There are two ways to determine convergence. First, one can do a substitution $u = \ln x$, $x = e^u$, $dx = e^u\, du$, to convert the integral to $\int_0^1 \frac{e^u}{u}\, du$. This diverges since $\int_0^1 \frac{du}{u}$ diverges and $\frac{e^u}{u} \geq \frac{1}{u}$ for $u \geq 0$. The other way is to compare $\int_1^e \frac{dx}{\ln x}$ directly with $\int_1^e \frac{dx}{x-1}$. By looking at the graphs of $y = \ln x$ and $y = x - 1$ one can see that $\ln x \leq x - 1$ for all x, so $1/\ln x \geq 1/(x-1)$ for all $x > 1$, hence $\int_1^e \frac{dx}{\ln x}$ diverges because $\int_1^e \frac{dx}{x-1}$ diverges.

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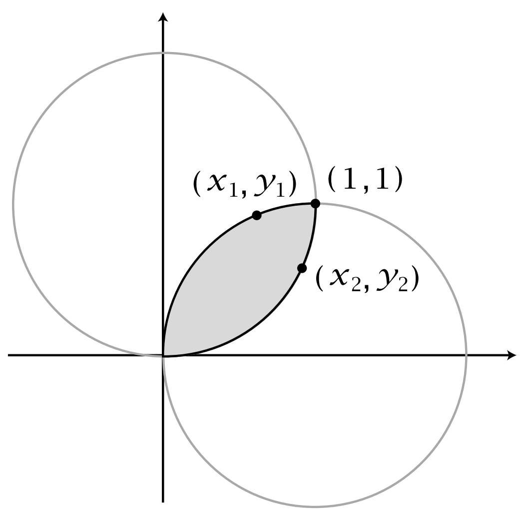
3. Consider the solid obtained by rotating the region that lies inside both the circles $(x-1)^2 + y^2 = 1$ and $x^2 + (y-1)^2 = 1$ about the x-axis.

(a) Set up integrals which compute the volume of this solid by both the slice method and the cylindrical shells method.

Here is what the region looks like, where we use coordinates (x_1, y_1) on one circle and coordinates (x_2, y_2) on the other:



For the upper boundary arc of the region we have

$$(x_1 - 1)^2 + y_1^2 = 1 \qquad y_1 = \sqrt{1 - (x_1 - 1)^2} \qquad x_1 = 1 - \sqrt{1 - y_1^2}$$

where this last square root has a minus sign in front of it because the x_1 coordinate of points on this arc has to be less than 1. For the lower boundary arc of the region we have similar formulas

$$x_2^2 + (y_2 - 1)^2 = 1 \qquad x_2 = \sqrt{1 - (y_2 - 1)^2} \qquad y_2 = 1 - \sqrt{1 - x_2^2}$$

Now we can write down integrals for the volume. By the slice method we get

$$\pi \int_0^1 (y_1^2 - y_2^2)\,dx = \pi \int_0^1 \left[1 - (x-1)^2 - (1 - \sqrt{1-x^2})^2\right] dx$$

and by the shell method we get

$$2\pi \int_0^1 (x_2 - x_1) y\,dy = 2\pi \int_0^1 \left[\sqrt{1-(y-1)^2} - (1 - \sqrt{1-y^2})\right] y\,dy$$

(b) Evaluate one of the integrals in part (a), whichever one you prefer.

The integral for the slice method is easier to compute:

$$\begin{aligned} &\pi \int_0^1 \left[1 - (x-1)^2 - (1-\sqrt{1-x^2})^2\right] dx \\ &= \pi \int_0^1 \left[1 - (x^2 - 2x + 1) - (1 - 2\sqrt{1-x^2} + 1 - x^2\right] dx \\ &= \pi \int_0^1 (2x - 2)\,dx + \pi \int_0^1 2\sqrt{1-x^2}\,dx \end{aligned}$$

Notice that the integral $\int_0^1 \sqrt{1-x^2}\,dx$ is equal to one quarter of the area of a circle of radius 1, so it equals $\pi/4$. Thus we get

$$\pi\left(x^2 - 2x\right)\Big|_0^1 + 2\pi\left(\frac{\pi}{4}\right) = -\pi + \frac{\pi^2}{2}$$

If you didn't happen to notice the easy way to compute $\int_0^1 \sqrt{1-x^2}\,dx$, you can compute it directly by substituting $x = \sin\theta$, $dx = \cos\theta\,d\theta$, to get

$$\int_0^{\pi/2} \cos^2\theta\,d\theta = \frac{1}{2}\int_0^{\pi/2} (1 + \cos 2\theta)\,d\theta = \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/2} = \frac{\pi}{4}$$