(30 pts) **1.** Compute the following integrals:

(a) $\displaystyle\int \frac{x^4+3x^3+3x^2+x+1}{x^4+x^2}\,dx$

Solution. First do long division to write $\dfrac{x^4+3x^3+3x^2+x+1}{x^4+x^2}$ as $1+\dfrac{x^3+2x^2+x+1}{x^4+x^2}$. The denominator factors as $x^2(x^2+1)$ so we can use partial fractions to write

$$\begin{aligned}\frac{x^3+2x^2+x+1}{x^2(x^2+1)} &= \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1}\\ &= \frac{Ax(x^2+1)+B(x^2+1)+x^2(Cx+D)}{x^2(x^2+1)}\end{aligned}$$

Comparing the coefficients of $x^2, x^2, x, 1$ gives the equations $A+C=3$, $B+D=2$, $A=1$, $B=1$. Plugging the values for A and B back into the first two equations gives $C=2$ and $D=1$. The integral is now

$$\int\left(1+\frac{1}{x}+\frac{1}{x^2}+\frac{2x}{x^2+1}+\frac{1}{x^2+1}\right)dx = x+\ln x - x^{-1}+\ln(x^2+1)+\tan^{-1}x+C$$

(b) $\displaystyle\int_0^2 \frac{dx}{(4+x^2)^{5/2}}$

Solution. Let $x=2\tan\theta$ so $dx = 2\sec^2\theta\,d\theta$. Then the integral becomes

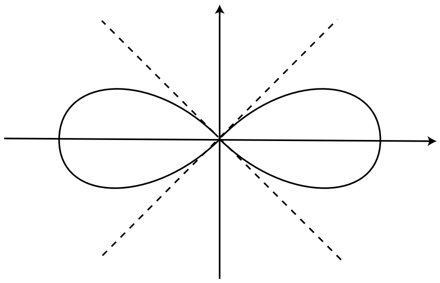
$$\begin{aligned}\int_0^{\pi/4}\frac{2\sec^2\theta}{(4\sec^2\theta)^{5/2}}\,d\theta &= \frac{1}{16}\int_0^{\pi/4}\cos^3\theta\,d\theta\\ &= \frac{1}{16}\int_0^{\pi/4}\left(\cos\theta-\cos\theta\sin^2\theta\right)d\theta = \frac{1}{16}\left(\sin\theta-\frac{\sin^3\theta}{3}\right)\Big|_0^{\pi/4}\\ &= \frac{1}{16}\left(\frac{\sqrt{2}}{2}-\frac{2\sqrt{2}}{8\cdot 3}\right) = \frac{5\sqrt{2}}{12\cdot 16}\end{aligned}$$

(c) $\displaystyle\int \ln\sqrt{1+x^2}\,dx$

Solution. Use integration by parts. We can let $u=\ln\sqrt{1+x^2}$ and $dv = dx$, so $v=x$ and $du = \dfrac{1}{\sqrt{1+x^2}}\cdot\dfrac{x}{\sqrt{1+x^2}}\,dx$. Then the integral becomes

$$\begin{aligned}x\ln\sqrt{1+x^2}-\int\frac{x^2}{1+x^2}\,dx &= x\ln\sqrt{1+x^2}-\int\left(1-\frac{1}{x^2+1}\right)dx\\ &= x\ln\sqrt{1+x^2}-x+\tan^{-1}x+C\end{aligned}$$

(15 pts) **2.** (a) Sketch the curve $r^2 = \cos 2\theta$ in polar coordinates and compute the area inside the curve.



Solution. Since r^2 is always ≥ 0, there will only be points on the curve when $\cos 2\theta \geq 0$, which means $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ and $\frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}$. For the area inside the curve we use the formula $\frac{1}{2}\int r^2 \, d\theta$. By symmetry we can integrate from 0 to $\pi/4$ and then multiply by 4. Thus we get

$$2\int_0^{\pi/4} \cos 2\theta \, d\theta = \sin 2\theta \Big|_0^{\pi/4} = 1$$

(b) Compute the length of the curve parametrized by $x = \cos^3 t$, $y = \sin^3 t$, for $0 \leq t \leq 2\pi$.

Solution. The arclength is

$$\begin{aligned}
\int_0^{2\pi} \sqrt{(dx/dt)^2 + (dy/dt)^2} \, dt &= \int_0^{2\pi} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} \, dt \\
&= 3\int_0^{2\pi} \sqrt{\sin^2 t \cos^2 t} \, dt \\
&= 3\int_0^{2\pi} |\sin t \cos t| \, dt \\
&= 12\int_0^{\pi/2} \sin t \cos t \, dt = 6\sin^2 t \Big|_0^{\pi/2} = 6
\end{aligned}$$

(10 pts) **3.** Consider the region between the curve $y = \sin x$ and the x-axis for $0 \leq x \leq 2\pi$. Compute the volume of the solid obtained by rotating this region about the y-axis.

Solution. Using cylindrical shells, the volume would be $2\pi\int_0^{2\pi} xy \, dx = 2\pi\int_0^{2\pi} x\sin x \, dx$. However, the function $\sin x$ is negative in the interval $[\pi, 2\pi]$ so the integral would be negative on this interval, which is not what we want. So we integrate from 0 to π with a plus sign and from π to 2π with a minus sign:

$$2\pi\int_0^{\pi} x\sin x \, dx \; - \; 2\pi\int_{\pi}^{2\pi} x\sin x \, dx$$

Using integration by parts with $u = x$ and $dv = \sin x$, hence $du = dx$ and $v = -\cos x$, we get

$$2\pi\big(-x\cos x + \sin x\big)\Big|_0^{\pi} \; - \; 2\pi\big(-x\cos x + \sin x\big)\Big|_{\pi}^{2\pi} = 2\pi(\pi) + 2\pi(2\pi + \pi) = 8\pi^2$$

(15 pts) **4.** Determine whether each of the following series converges:

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$. Use the integral test: $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = -(\ln x)^{-1}\Big|_2^{\infty}$ which is finite so the series converges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{2n^2+1}$. The individual terms have absolute value $\dfrac{n^2}{2n^2+1} \to \dfrac{1}{2}$, so the terms do not approach zero and the series diverges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$. Using the ratio test, we look at $\dfrac{(n+1)!}{(n+1)^{n+1}} \cdot \dfrac{n^n}{n!} = \dfrac{(n+1)n^n}{(n+1)^{n+1}} = \dfrac{n^n}{(n+1)^n}$. To compute the limit as $n \to \infty$ we take the logarithm: $\ln\Big(\dfrac{n}{n+1}\Big)^n = n\big(\ln n - \ln(n+1)\big)$. This has the indeterminate form $\infty \cdot 0$ so we apply l'Hopital's rule, first rewriting it as $\dfrac{\ln n - \ln(n+1)}{n^{-1}}$. Then taking derivatives of numerator and denominator gives

$$\frac{\frac{1}{n} - \frac{1}{n+1}}{-\frac{1}{n^2}} = \frac{\frac{(n+1)-n}{n(n+1)}}{-\frac{1}{n^2}} = \frac{-n^2}{n(n+1)} \quad \to \quad -1$$

Thus $\displaystyle\lim_{n\to\infty}\Big(\frac{n}{n+1}\Big)^n = e^{-1}$. This is less than 1 so the series converges.

(d) Determine the values of $x > 0$ for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{x^n + x^{n-1} + \cdots + x + 1}$ converges.

Solution. There are $n+1$ terms in the denominator so when $x = 1$ the series becomes the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+1}$ which diverges. When $n < 1$ we have $x^n + x^{n-1} + \cdots + x + 1 < n+1$ so $\dfrac{1}{x^n + x^{n-1} + \cdots + x + 1} > \dfrac{1}{n+1}$ and the series diverges by comparison with the harmonic series. When $x > 1$ we compare the series with the geometric series $\sum \dfrac{1}{x^n}$. We have $x^n + x^{n-1} + \cdots + x + 1 > x^n$ so $\dfrac{1}{x^n + x^{n-1} + \cdots + x + 1} < \dfrac{1}{x^n}$. Since $x > 1$ the geometric series $\sum \dfrac{1}{x^n}$ converges, hence the given series also converges when $x > 1$.

(20 pts) **5.** (a) Determine the values of x for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n (x-1)^{2n}}{n^2}$ converges.

Solution. Ratio test: $\dfrac{2^{n+1}|x-1|^{2(n+1)}}{(n+1)^2} \cdot \dfrac{n^2}{2^n |x-1|^{2n}} = 2|x-1|^2 \dfrac{n^2}{(n+1)^2} \to 2|x-1|^2$. Thus the series converges if $|x-1|^2 < 1/2$, or equivalently $|x-1| < 1/\sqrt{2}$, or $1 - \sqrt{2} < x < 1 + \sqrt{2}$. At the endpoints of the interval we have $(x-1)^2 = 1/2$ so the series becomes

$\sum \frac{1}{n^2}$ which converges, being a p-series with $p > 1$. Thus the interval of convergence is $1 - \sqrt{2} \leq x \leq 1 + \sqrt{2}$.

(b) Compute the Maclaurin series for the function $f(x) = \dfrac{x^2}{(1-x)^2}$.

Solution. Let $g(x) = \dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, so $g'(x) = \dfrac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}$. Multiplying this by x^2, we get $\dfrac{x^2}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n+1}$.

(c) Find the Taylor series $\sum_{n=0}^{\infty} a_n(x-2)^n$ for $\ln x$ and determine the interval of convergence.

Solution. First we will find a similar series for $1/x$, then integrate this to get a series for $\ln x$. We have

$$\frac{1}{x} = \frac{1}{2+(x-2)} = \frac{1}{2}\Big(\frac{1}{1+(\frac{x-2}{2})}\Big) = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(x-2)^n}{2^n}$$

Integrating this, we get

$$\ln x = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(x-2)^{n+1}}{(n+1)2^n} \ + \ C$$

We determine C by setting $x = 2$, which makes the series have the value 0 so $C = \ln 2$. To determine the interval of convergence we use the ratio test:

$$\frac{|x-2|^{n+2}}{(n+2)2^{n+1}} \cdot \frac{(n+1)2^n}{|x-2|^{n+1}} \quad \to \quad \frac{|x-2|}{2}$$

Thus the series converges when $|x-2|/2 < 1$, or $|x-2| < 2$, which means $0 < x < 4$. At the endpoint $x = 0$ we have the series $-\frac{1}{2}\sum \frac{1}{n+1}$ which diverges. At $x = 4$ we have $\frac{1}{2}\sum(-1)^n \frac{1}{n+1}$ which converges by the alternating series test. The interval of convergence is therefore $0 < x \leq 4$.

(d) Suppose the ratio test yields the information that a certain series $\sum_{n-0}^{\infty} a_n(x-c)^n$ has radius of convergence equal to R. Show that the derivative series $\sum_{n-0}^{\infty} na_n(x-c)^{n-1}$ has the same radius of convergence R.

Solution. To apply the ratio test to the series $\sum_{n-0}^{\infty} a_n(x-c)^n$ we compute the limit $\lim_{n\to\infty} \dfrac{a_{n+1}|x-c|^{n+1}}{a_n|x-c|^n} = |x-c| \lim_{n\to\infty} \dfrac{a_{n+1}}{a_n}$. For the derivative series $\sum_{n-0}^{\infty} na_n(x-c)^{n-1}$ the corresponding limit is $\lim\limits_{n\to\infty} \dfrac{(n+1)a_{n+1}|x-c|^{n+1}}{na_n|x-c|^n} = |x-c| \lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$. This is the same limit as before, so the radius of convergence is the same.

(10 pts) **6.** (a) Find the general solution of the differential equation $x^2y' - y = 1$.

Solution. There are two methods that work: separation of variables, and finding an integrating factor $\mu(x)$. Let's use the latter method. First rewrite the equation in the form $y' - x^{-2}y = x^{-2}$, so $P(x) = -x^{-2}$ and $\mu(x) = e^{\int P\,dx} = e^{\int -x^{-2}\,dx} = e^{x^{-1}}$. After multiplying the equation $y' - x^{-2}y = x^{-2}$ by $\mu(x)$ we get $\left(e^{x^{-1}}y\right)' = e^{x^{-1}}x^{-2}$. Integrating both sides, we get $e^{x^{-1}}y = -e^{x^{-1}} + C$. Solving this for y gives $y = -1 + Ce^{-x^{-1}}$.

(b) Find the solution of $y'' + 2y' + 2y = 0$ satisfying $y(0) = 1$ and $y'(0) = 0$.

Solution. The auxiliary equation is $r^2 + 2r + 2 = 0$ with roots $r = (-2 \pm \sqrt{-4})/2 = -1 \pm i$ so the general solution is $y = Ae^{-x}\cos x + Be^{-x}\sin x$. Plugging in the condition $y(0) = 1$ gives $A = 1$. For the condition $y'(0) = 0$ we first compute $y' = A(e^{-x}(-\sin x) - e^{-x}\cos x) + B(e^{-x}\cos x - e^{-x}\sin x)$. Then $y'(0) = -A + B = 0$ so $B = A = 1$. The final answer is $y = e^{-x}(\cos x + \sin x)$.

(c) Find a differential equation satisfied by all of the hyperbolas $xy = C$, for C an arbitrary constant.

Solution. Differentiating $xy = C$ implicitly gives $x(dy/dx) + y = 0$, so $dy/dx = -y/x$.

(d) Find a family of curves, each of which intersects each hyperbola in part (c) at right angles.

Solution. The slope of these curves will be the negative reciprocal of the slopes $-y/x$ in part (c). Thus we have the differential equation $dy/dx = x/y$. This can be rewritten as $y\,dy = x\,dx$. Integrating this, we get $\int y\,dy = \int x\,dx$, or $y^2/2 = x^2/2 + C$. This can also be written as $y^2 - x^2 = C$ for a new constant C.