Solutions, Prelim 1

September 25, 2017

Q 1

Question: Let \((F, +, \cdot, 0, 1, \leq)\) be an ordered field. Prove that for every \(a \in F\),
1. if \(a \neq 0\) then \(a^{-1} \neq 0\);
2. if \(a > 0\) then \(a^{-1} > 0\).

Solution:

(1) Since \(a \neq 0\), there is \(a^{-1} \in F\) such that \(aa^{-1} = 1\). If \(a^{-1} = 0\), then
\[
0 = a \cdot 0 = a \cdot a^{-1} = 1,
\]
contradicting our assumption that \(0 \neq 1\). We conclude that \(a^{-1} \neq 0\).

(2) Assume that \(a > 0\). By (1), \(a^{-1} \neq 0\), so \(a^{-1}a^{-1} > 0\) (recall Homework 1).
Hence, by an axiom of ordered fields,
\[
a \cdot (a^{-1}a^{-1}) > a \cdot 0.
\]
Thus, we see that \(a^{-1} > 0\).

Q 2

Question:

1. **True or False:** The set \(\mathbb{Q}\) of rational numbers, with the usual order relation, satisfies the least upper bound property.

2. **True or False:** The set \(\mathbb{N}\) of natural numbers, with the usual order relation, satisfies the greatest lower bound property.
Solution:

(1) True. Let \( S \subseteq \mathbb{N} \) be non-empty. Then \( S \) has a least element \( x_0 \in S \). If a set has a least element, then this least element is the greatest lower bound for the set.

Q 3

Question: True or False? Shortly explain your answers. If true, justify; if false, give a counter example.

1. If a sequence \((a_n)\) is convergent than the sequence \((|a_n|)\) is convergent.
2. An increasing sequence of negative numbers must have a limit.
3. A decreasing sequence of negative numbers must have a limit.
4. If a sequence \((a_n)\) is such that \(\lim_{n \to \infty} (a_{n+1} - a_n) = 0\) then \((a_n)\) is convergent.

Solution:

(1) False. The alternating sequence \( a_n = (-1)^n \) is not convergent though the sequence \((|a_n|) = (1, 1, 1, \ldots)\) does converge (to 1).

(2) True. If the sequence \((a_n)\) is increasing and \( a_n \leq 0 \) for all \( n \), then \((a_n)\) is bounded from above by 0, and any increasing sequence of real numbers bounded above converges, and the limit is

\[
\sup\{a_n | n \in \mathbb{N}\}
\]

by the monotone convergence theorem.

(3) False. Define

\[
a_n = -n, \quad n \in \mathbb{N}.
\]

Then \((a_n)\) is a decreasing sequence of negative numbers diverging to infinity, in the sense that for any \( M < 0 \), there is an \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \), \( a_n < M \).

(4) False. The harmonic series \( a_n = \sum_{k=1}^{n} \frac{1}{k} \) is divergent but \( a_{n+1} - a_n = \frac{1}{n+1} \) converges to 0.
Q4

**Question:**

1. Assume that $a_n$ are all non-negative. Show that if $\sum_{n=1}^{\infty} a_n$ converges then so does $\sum_{n=1}^{\infty} a_n^2$.

2. Give an example of $(a_n)$ such that $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} a_n^2$ diverges. Justify your answer.

**Solution:**

(1) Since $\sum_{n=1}^{\infty} a_n$ converges, the term $a_n$ converges to zero. In particular, (for $\epsilon = 1$), there is $N \in \mathbb{N}$ such that for $n \geq N$, we have $|a_n| = |a_n - 0| < 1$. Since $a_n \geq 0$ for all $n$, we deduce that for $n > N$, $0 \leq a_n^2 \leq a_n$. Therefore, by the comparison theorem, the convergence of $\sum_{n=1}^{\infty} a_n$ implies the convergence of $\sum_{n=1}^{\infty} a_n^2$.

In details: we deduce that since the sequence of partial sums $\sum_{n=1}^{\infty} a_n$ is bounded (as a convergent sequence), so is the sequence of partial sums $\sum_{n=1}^{\infty} a_n^2$. That sequence is also increasing, since $a_n^2 \geq 0$ for all $n$. Therefore, by the monotone convergence theorem, $\sum_{n=1}^{\infty} a_n^2$ is convergent.

(2) For $a_n = \frac{(-1)^n}{\sqrt{n}}$ the series $\sum_{n=1}^{\infty} a_n$ converges, by the alternating series theorem, since $(|a_n|)$ is a decreasing sequence that converges to 0. The series $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series and is divergent.

Q5

**Question:**

1. Show that the function $f(x) = x^3 + 2\sin(x)$ has a root in $\mathbb{R}$ (i.e., a value $x_0$ such that $f(x_0) = 0$).

2. Give an example of a function $f(x)$ and a real number $a$ such that there is a sequence $(a_n)$ with $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} f(a_n) = f(a)$ but $f$ is not continuous at $a$. Justify your answer.

**Solution:**

1. This question turned out to be easier than planned since $f(0) = 0$. Here is a more general solution. Since $f$ is continuous on $[-\pi, \pi]$, as a sum of continuous functions: a polynomial and a constant times $\sin(x)$, we deduce from the Intermediate Value Theorem that $f$ takes all values between $f(-\pi)$ and $f(\pi)$. In particular, since $f(-\pi) < 0 < f(\pi)$, there is $b \in [-\pi, \pi]$ such that $0 = f(b)$. 

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2. The function \( f(x) = \begin{cases} 1 & x \notin \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases} \) defined in Problem (3) in Extra 9/19 is a good example. For every sequence of rational numbers that converges to 0 (e.g., \( a_n = \frac{1}{n} \)) the sequence \( (f(a_n)) \) converges to \( f(0) = 0 \). However \( f \) is not continuous since there is a sequence \( b_n \) of irrational numbers that converges to 0 (e.g., \( b_n = \frac{1}{n^2} + \pi \)) with \( (f(b_n)) \) that converges to \( 1 \neq f(0) \).

**Bonus**

*Question:* Show that for every real number \( a \in \mathbb{R} \) with \( a > 0 \) there is an \( n \in \mathbb{N} \) such that \( n - 1 \leq a < n \).

*Solution:* Let \( a \in \mathbb{R} \) with \( a > 0 \). Define a subset of \( \mathbb{N} \) by

\[
S = \{ m \in \mathbb{N} : a < m \}.
\]

By the Archimedian property of \( \mathbb{R} \), it follows that \( S \neq \emptyset \). By Q 2. 2), \( S \) has a greatest lower bound \( n_0 \in \mathbb{N} \). Since \( n_0 \in S \), it follows that

\[
a < n_0.
\]

Since \( n_0 \) is a lower bound for \( S \), \( n_0 - 1 \notin S \). Therefore,

\[
a \geq n_0 - 1.
\]

In total,

\[
n_0 - 1 \leq a < n_0,
\]

as desired.