Remember - We don’t expect that everyone will solve every problem, but we do expect that everyone make a serious attempt at every problem and explain what you tried when you can’t solve a problem.

Math 1220, Fall 2017

Assignment due 8/31 in Lecture

1. Let \((F, +, \cdot, 0, 1)\) be a field. Prove that for every elements \(a, b\) of \(F\) the following claims hold.

   (a) \(a + a = a\) implies \(a = 0\).

   (b) There is a unique element \(a'\) of \(F\) such that \(a' + a = 0\). We denote this element by \(-a\).

   (c) \(a \cdot 0 = 0\).

   (d) \((-a) \cdot b = -(a \cdot b); (-a) \cdot (-b) = a \cdot b\).

   (e) \(a \cdot b = 0\) implies \(a = 0\) or \(b = 0\).

2. Let \((F, +, \cdot, 0, 1, \leq)\) be an ordered field. We write \(a < b\) to denote that \(a \leq b\) and \(a \neq b\). We write \(a^2\) for \(a \cdot a\). Prove the following claims.

   • For every element \(a\) of \(F\), it holds that \(0 \leq a^2\), and if \(a \neq 0\) then \(0 < a^2\).

   • \(0 < 1\) and \(-1 < 0\).

3. Show that if the equation \(x^2 + 1 = 0\) has a solution in a field \(F\), then \(F\) cannot be an ordered field, no matter how \(\leq\) is defined.

4. • Show that every ordered field \(F\) contains (a copy of the) natural numbers. (Hint: show, by induction, that the elements \(0, 1, 1 + 1, 1 + 1 + 1, \ldots\) are all different.)

   • We say that an ordered field \(F\) satisfies the Archimedean property if for every element \(a\) of \(F\) there is a natural number \(n\) such that \(a < n\). Show that if \(F\) is a complete ordered field then it satisfies the Archimedean property.