Remember - We don’t expect that everyone will solve every problem, but we do expect that everyone make a serious attempt at every problem and explain what you tried when you can’t solve a problem.

Math 1220, Fall 2017

(1) Proofs by mathematical induction of a family of statements $P_n$ (a natural number) involve proving a base case (e.g. the $n = 1$ version) and then proving the truth of $P_n$ implies the truth of $P_{n+1}$. And then one concludes the statements are true for all $n$ greater than or equal to the “n” in the base case.

This problem asks you to show that all higher order derivatives of

$$f(x) = \begin{cases} 
e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

exist at $x = 0$ and are 0.

a: Show using induction for $x > 0$ and any $n \geq 0$ that $f^{(n)}(x)$ is of the form $P_n(\frac{1}{x})e^{-\frac{1}{x}}$ for some polynomial $P$.
b: Show that $\lim_{x \to 0} P_n(\frac{1}{x})e^{-\frac{1}{x}} = 0$.
c: Prove that $f^{(n)}(0) = 0$. (Note that this completes the proof that the Taylor series of $f$ about $x = 0$ has radius of convergence infinity, but for $x > 0$, never converges to $f(x)!$)

(2) a: Find $\lim_{n \to \infty} n^{\frac{1}{n}}$.
b: Prove that if the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$, then for any $|x| < R$, $\sum_{n=0}^{\infty} n a_n x^{n-1}$ also converges. (So you’ve proven the term by term differentiated series has at least as big a radius of convergence.)
Let $\epsilon > 0$ be given. Let $f_n(x) : \mathbb{R} \to \mathbb{R}$ be sequence of functions converging uniformly to $f(x) : \mathbb{R} \to \mathbb{R}$. We suppose each $f_n$ is differentiable with $f_n'$ continuous and that $f_n'(x)$ converges uniformly to $g(x) : \mathbb{R} \to \mathbb{R}$. Prove $f'(x) = g(x)$ as follows:

a) Fix $x_0 \in \mathbb{R}$. Since $g$ is the uniform limit of a sequence of continuous functions, $g$ is continuous at $x$. So for $|h|$ small enough we have $|g(x_0 + h) - g(x_0)| < \epsilon$. There’s nothing for you to prove here.

b) Here $h$ will be small enough so that $|g(x_0 + h) - g(x_0)| < \epsilon$ as in a). Estimate $\left| f(x_0 + h) - f(x_0) \right| h$ by approximating $g(x_0)$ by $f_n'(x_0)$ (with an error of at most $\epsilon$) and $f_n(x_0 + h)$ and $f_n(x_0)$ with errors of at most $\epsilon h$ using the uniform convergences. In both cases, $n$ must be big enough. So

$$\left| f(x_0 + h) - f(x_0) \right| h \leq 3\epsilon + \left| f_n(x_0 + h) - f_n(x_0) - f_n'(x_0) \right| .$$

c) Use the Mean Value Theorem to write $\left| f_n(x_0 + h) - f_n(x_0) \right| = f_n'(c)$ for some $c$ between $x$ and $x + h$ (remember, $h$ might be negative).

d) Now use uniform convergence to show $\left| f_n'(c) - f_n'(x_0) \right| \leq 2\epsilon + |g(c) - g(x_0)|$.

e) Show $f'(x_0) = g(x)$.

Using 3) show that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$ then $f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$ as follows:

a) Using 2) above, if $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$ then $\sum_{n=0}^{\infty} n a_n x^{n-1}$ has radius of convergence at least $R$ and hence these power series converge uniformly on closed intervals $[-S, S]$ where $S < R$. Set $f_m(x) = \sum_{n=0}^{m} a_n x^n$. What is $f'_m(x)$?

b) Since $f_m(x)$ and $f'_m(x)$ converge uniformly on $[-S, S]$, their limits are continuous. Use 3) to get the result. We like this proof better than the one in the book. Which do you like better?