1. (12 points) According to the state of California, 33% of all state community college students belong to ethnic minorities. Find the probabilities of the following results in a random sample of 10 California community college students. Do *not* simplify your answers.
   Solution. For all three parts, we need to use the formula for the probability of k successes in n trials of a binomial experiment.

   (a) Exactly 2 belong to an ethnic minority.
   Solution. The probability that exactly 2 belong to an ethnic minority is

$$\binom{10}{2} \cdot \left(\frac{1}{3}\right)^2 \cdot \left(\frac{2}{3}\right)^8$$

   (b) Three or fewer belong to an ethnic minority.
   Solution. The probability that exactly 3 or fewer belong to an ethnic minority is

$$\begin{aligned} &\binom{10}{0} \cdot \left(\frac{1}{3}\right)^0 \cdot \left(\frac{2}{3}\right)^{10} \\ +&\binom{10}{1} \cdot \left(\frac{1}{3}\right)^1 \cdot \left(\frac{2}{3}\right)^9 \\ +&\binom{10}{2} \cdot \left(\frac{1}{3}\right)^2 \cdot \left(\frac{2}{3}\right)^8 \\ +&\binom{10}{3} \cdot \left(\frac{1}{3}\right)^3 \cdot \left(\frac{2}{3}\right)^7 \end{aligned}$$

   (c) Exactly 5 do not belong to an ethnic minority.
   Solution. The probability that exactly 5 do not belong to an ethnic minority is the same as the probability that exactly 10-5 = 5 *do* belong to an ethnic minority, which is

$$\binom{10}{5} \cdot \left(\frac{1}{3}\right)^5 \cdot \left(\frac{2}{3}\right)^5$$

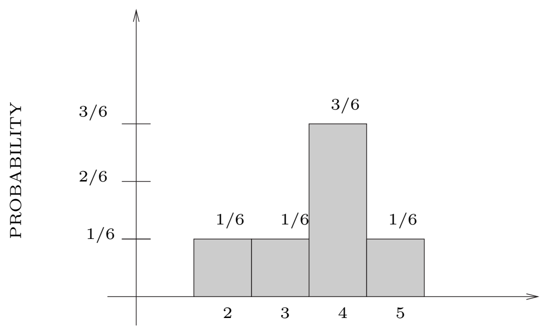
   Note: this is very different than $1 - \binom{10}{5} \cdot \left(\frac{1}{3}\right)^5 \cdot \left(\frac{2}{3}\right)^5$! This second expression is the probability that *any number other than 5* students belong to an ethnic minority.

2. (20 points) Instead of the typical sides numbered one through six, an unusual six-sided die has 1 side with the number two, 1 side with the number three, 3 sides with the number four, and 1 side with the number five.

   (a) Draw a histogram for the probability distribution for the possible outcomes of rolling this die.
   Solution. The probability distribution is

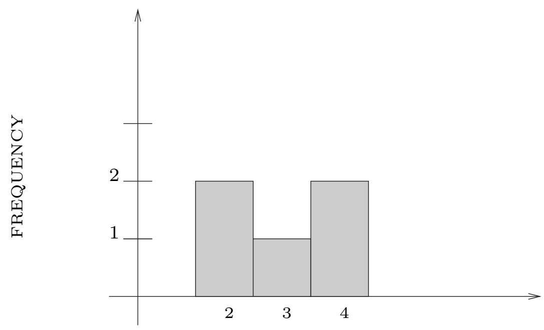
x	2	3	4	5
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{3}{6}$	$\frac{1}{6}$

   and so the histogram for this distribution is



   (b) What is the expected value of a roll of this die?
   Solution. The expected value is $E(x) = \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{3}{6} \cdot 4 + \frac{1}{6} \cdot 5 = \frac{22}{6} = \frac{11}{3}$.

   (c) The die was rolled 5 times giving the outcomes 2, 2, 3, 4, and 4. Draw the histogram for this frequency distribution.
   Solution.



(d) Determine the mean, median, mode(s) and standard deviation of this frequency distribution.
Solution. We are given $n = 5$ data with

x_1	x_2	x_3	x_4	x_5
2	2	3	4	4

The mean is

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_5}{n} = \frac{2+2+3+4+4}{5} = \frac{15}{5} = 3.$$

The median is clearly 3, and since both 2 and 3 occur twice, the modes are 2 and 4. Finally, the standard deviation is

$$\begin{aligned}\sigma &= \sqrt{\frac{(x_1-\bar{x})^2 + (x_2-\bar{x})^2 + \cdots + (x_5-\bar{x})^2}{n-1}} \\ &= \sqrt{\frac{(2-3)^2 + (2-3)^2 + (3-3)^2 + (4-3)^2 + (4-3)^2}{4}} \\ &= \sqrt{\frac{1+1+0+1+1}{4}} \\ &= 1.\end{aligned}$$

3. (16 points) The chickens at Colonel Thompson's Ranch have a mean weight of 1850 grams, with a standard deviation of 150 grams. The weights of the chickens are closely approximated by a normal curve.

(a) Find the percent of all chickens having weights in the following ranges.

(i) More than 1700 grams.
Solution. 1700g has a z value of -1, since

$$z = \frac{x-\mu}{\sigma} = \frac{1700-1850}{150} = -1$$

We want to know the area to the right of $z = -1$ under the standard normal curve. Since the table tells us that the area to the *left* of $z = -1$ is 0.1587, the area to the right must be $1 - 0.1587 = 0.8413$. Thus, there is about an 84% chance that a chicken weighs at least 1700g.

(ii) Less than 1950 grams.
Solution. 1950 has a z value of 2/3 since

$$\frac{1950-1850}{150}=\frac{100}{150}=2/3$$

Unfortunately, the table only gives us results for negative z, so we cannot directly look up $z = 2/3$. However, since the normal curve is symmetric, the area to the *left* of $z = 2/3$ is the same as the area to the *right* of $z = -2/3$. Since the area to the left of $z = -2/3$ is 0.2514, the area we want is $1 - 0.2514 = 0.7486$. Therefore, there is about a 75% chance that a chicken weighs less than 1950g.

(iii) Between 1750 grams and 1900 grams.
Solution. There are several ways to approach this problem. For this solution, we will start with 1 (the area under the whole normal curve) and subtract the areas of the two tails, below 1750g and above 1900g. The area below 1750 is easy to find. 1750 has a z value of $-2/3$:

$$z=\frac{1750-1850}{150}=-100150=-2/3$$

so, from the table, we know that the area of this tail is 0.2514. To find the area above 1900g, first find the z value

$$z=\frac{1900-1850}{150}=1/3$$

Then the area above $z = 1/3$ is the same as the area below $z = -1/3$, so the area of this tail is 0.3707. So the area between 1750g and 1900g is $1 - 0.2514 - 0.3707 = 1 - 0.6221 = 0.3779$. That is, there is about a 38% chance of being in this range of weights.

(b) Helen Hen weighs more than approximately 97.5% of the other chickens at Colonel Thompson's Ranch. How much does Helen weigh?
Solution. Since Helen Hen weighs more than 97.5% of the other hens, Helen's weight has a z value of 1.96 (found using the table and the fact that $1 - 0.9750 = 0.0250$). Then if x is Helen's weight, we know that

$$\begin{aligned} 1.96 &= \frac{x-1850}{150} \\ 150\cdot 1.96+1850 &= x \\ 2144 &= x \end{aligned}$$

so Helen Hen must weigh 2144g.

4. (16 points) The National Institute of Health estimates that 10 percent of all Americans 20 years of age or older have diabetes.

   (a) A sample of 100 Americans 20 years of age or older is taken. Use binomial probability only to find *exactly* the probability that

      (i) exactly 16 will have diabetes. Do *not* simplify your answer.
      Solution. $\binom{100}{16}(.1)^{16}(.9)^{84}$

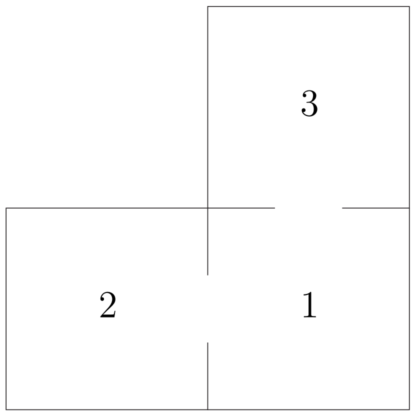
      (ii) more than 16 of the Americans sampled have diabetes. Do *not* simplify your answer.
      Solution. $\binom{100}{17}(.1)^{17}(.9)^{83} + \binom{100}{18}(.1)^{18}(.9)^{82} + \ldots + \binom{100}{99}(.1)^{99}(.9)^{1} + \binom{100}{100}(.1)^{100}(.9)^{0}$

   (b) Use the normal approximation to the binomial probability of part (a) and the table on the last page to *approximate* the probability that

      (i) more than 16 of the Americans sampled have diabetes. Simplify your answer.
      Solution. The mean and standard deviation of the approximating normal curve are $\mu = 100(.1) = 10$ and $\sigma = \sqrt{100(.1)(.9)} = 3$ respectively. The probability that more than 16 of the Americans sampled have diabetes is equal to the area under the normal curve that is to the right of the line $x = 16.5$. We translate to the standard normal curve by calculating the z-score of this value, $z = \frac{16.5-10}{3} = \frac{13}{6}$. By the symmetry of the normal curve with respect to the y-axis, the area under the standard normal curve to the right of $z = \frac{13}{6}$ is the same as the area to the left of $z = \frac{-13}{6}$ in the standard normal curve, which is .0150 by the table, and hence the probability is .015.

      (ii) exactly 16 of the Americans sampled have diabetes. Simplify your answer.
      Solution. The probability that exactly 16 Americans have diabetes is approximated by the area between $x = 15.5$ and $x = 16.5$ under the normal curve with mean μ and standard deviation σ. Translating to the standard normal curve, the z-score of 15.5 is $\frac{15.5-10}{3} = \frac{11}{6}$. The area between $x = 15.5$ and $x = 16.5$ is equal to the area between $z = \frac{11}{6}$ and $z = \frac{13}{6}$, which equals the area to the right of $z = \frac{11}{6}$ minus the area to the right of $\frac{13}{6}$. By the symmetry of the normal curve with respect to the y-axis, this area is equal to the area to the left of $z = \frac{-11}{6}$ minus the area to the left of $z = \frac{-13}{6}$, which is $.0344 - .0150 = .0194$. The approximate probability that exactly 16 will have diabetes is .0194.

5. (20 points) A man lives in a house with three rooms and never leaves his house. The only time that the man moves from one room to another is at the start of each hour, at which time the man decides whether or not to move into another room. Half of the time, he stays in the room he is in, and half of the time he chooses an adjoining room at random, and moves into that room. The figure below shows a blueprint of the man's house.



(a) What is the transition matrix of the Markov chain describing the man's position in the house?
Solution. The transition matrix is

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \end{array}$$

(b) Is this Markov chain regular? Justify your answer.
Solution. Yes the Markov chain is regular because P^2 has no zero entries:

$$P^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{2} & \frac{1}{8} & \frac{3}{8} \end{pmatrix}$$

(c) If the man is in room 1, what is the probability that he will be in room 3 two hours from now?
Solution. The probability is the entry in the 1st row and 3rd column of P^2, namely $\frac{1}{4}$.

(d) Does this Markov chain have an equilibrium vector? If so, find it. If not, explain why it does not exist.

Solution. Yes, Markov chain has an equilibrium vector since it is regular.

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{2} & \frac{1}{8} & \frac{3}{8} \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix}$$

yields the system of equations:

$$\begin{aligned} x + y + z &= 1 && (1) \\ \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z &= x && (2) \\ \frac{1}{4}x + \frac{1}{2}y &= y && (3) \\ \frac{1}{4}x + \frac{1}{2}z &= z && (4) \end{aligned}$$

Equation 3 gives us that $y = \frac{1}{2}x$ and Equation 4 gives us that $z = \frac{1}{2}x$. Substituting these results into Equation 1 yields that $x = \frac{1}{2}$, implying that $y = z = \frac{1}{4}$. Hence the equilibrium vector for this Markov chain is

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

6. (16 points) A man is playing two slot-machines. The first machine pays off with probability p and, the second with probability q. If he loses, he plays the same machine again; if he wins, he switches to the other machine. This describes a Markov chain with two states, "the man is playing the first machine" and "the man is playing the second machine".

   (a) What is the transition matrix P of this Markov chain?
   Solution. Let us denote the two states by

   $$s_1 = \text{"the man is playing the first machine"}$$

   and

   $$s_2 = \text{"the man is playing the second machine"}.$$

   Since the man plays the same machine if and only if he loses, the transition matrix of this Markov chain is

   $$P = \begin{array}{c} \\ s_1 \\ s_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} s_1 & \quad s_2 \end{array} \\ \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \end{array}$$

   (b) Suppose that for his first pull, the man plays the first machine. What is the probability, in terms of p and q, that for his third pull the man plays the second machine?
   Solution. The probability $p_{12}^{(2)}$ in question is the entry in the first row and second column of P^2. That is,

   $$p_{12}^{(2)} = \begin{pmatrix} 1-p & p \end{pmatrix} \begin{pmatrix} p \\ 1-q \end{pmatrix} = (1-p)p + p(1-q).$$

   (c) Suppose that the second machine never pays off.

   (i) Is this Markov chain regular? Justify your answer.
   Solution. No, it's not! Since the second machine never pays off, $q = 0$, and since the man switches machines if and only if he wins, the transition probability from s_2 to s_1 is 0 at any step. That is, $p_{21}^{(n)} = 0$ for all $n \geq 1$. Therefore, each power P^n of (the new) P has a zero entry in the second row and first column for all $n \geq 1$.

   (ii) Does this Markov chain have a probability vector V such that $VP = V$? If so, find such a V. If not, explain your answer.
   Solution. Since the second machine never pays off, and since the man switches machines if and only if he wins, the probability vector $V = \begin{bmatrix} 0 & 1 \end{bmatrix}$ satisfies $VP = V$ (check that!).

Useful approximate values

$$1/3 \approx 0.33 \quad \text{and} \quad 2/3 \approx 0.67.$$

Area under the standard normal curve

z	Area to left of z
-3	0.0013
-2.17	0.0150
-2	0.0228
-1.96	0.0250
-1.83	0.0344
-1.67	0.0475
-1.5	0.0668
-1	0.1587
-0.67	0.2514
-0.33	0.3707