1. (a) (4 points) Are there more license plates made of 3 letters, or made of 3 digits? Justify your answer.
   Solution. Choosing a license plate with three letters means picking one thing out of 26, repeated 3 times. So the total number of 3-letter license plates is $26 \cdot 26 \cdot 26 = 26^3$ and the number of 3-digit license plates is 10^3.

   (b) (4 points) How many license plates are there using 3 letters followed by 3 digits? Justify your answer.
   Solution. There are 26^3 3-letter license plates and 10^3 3-digit license plates, so the number of 3-letter-then-3-digit license plates is $26^3 \cdot 10^3$ by the multiplication principle.

   (c) (4 points) How many license plates are there using 3 letters followed by 3 digits with no repetition? Justify your answer.
   Solution. To pick 3 letters with out repetition you have 26 choices for the first letter, 25 for the second, and 24 for the third. Similarly, to pick 3 digits without repetition you have 10 choices for the first, 9 for the second, and 8 for the third. So, using the multiplication principle, the total number of 3-letter-then-3-digit-with-no-repetition license plates is $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 = P(26, 3) \cdot P(10, 3)$.

   You could also use combinations by first picking three letters of 26: $\binom{26}{3}$, then picking 3 digits of 10: $\binom{10}{3}$. Then we need to line those three letters up: 3! and line the three digits up: 3!. So the total number is $\binom{26}{3} \cdot \binom{10}{3} \cdot 3! \cdot 3!$.

2. (10 points) Three crows, 4 blue jays, and 5 starlings sit in a random order on a section of telephone wire. Find the probability that birds of a feather flock together, that is, that all birds of the same type are sitting together. Justify your answer.
   Solution. There are 3! ways that the groups of crows could line up (ex: {crows, jays, starlings}, {jays, crows, starlings}, etc). Then we need to line up the birds in each of the groups: 3! for the crows, 4! for the jays, and 5! for the starlings. So the total *number* of ways that birds of a feather could flock together is $3! \cdot 3! \cdot 4! \cdot 5!$. Since there are 12 birds, the total number of ways that they could line up is 12!, so the *probability* that birds of a feather are flocking together is

$$\frac{3! \cdot 3! \cdot 4! \cdot 5!}{12!}$$

3. (15 points) Consider the experiment where a six-sided die is rolled and a coin is flipped. Consider the following possible events in the sample space of this experiment:

   A: Two or less is rolled

   B: A tails is flipped

   C: A three is rolled or a heads is flipped

   Which pairs of these events are dependent? Which pairs of these events are independent? Explain your answer for each of the 3 pairs.

   Solution. The sample space of this experiment is:

$$\begin{aligned} S = \{ \quad & 1-H, \quad 2-H, \quad 3-H, \quad 4-H, \quad 5-H, \quad 6-H, \\ & 1-T, \quad 2-T, \quad 3-T, \quad 4-T, \quad 5-T, \quad 6-T \quad \} \end{aligned}$$

and

$$\begin{aligned} A &= \{\,1-H,\, 2-H,\, 1-T,\, 2-T\,\} \\ B &= \{\,1-T,\, 2-T,\, 3-T,\, 4-T,\, 5-T,\, 6-T\,\}, \\ C &= \{\,1-H,\, 2-H,\, 3-H,\, 4-H,\, 5-H,\, 6-H,\, 3-T\,\}. \end{aligned}$$

So

$$A \cap B = \{1-T, 2-T\}, \quad B \cap C = \{3-T\}, \quad \text{and} \quad A \cap C = \{1-H, 2-H\}$$

Since all outcomes of S are equally likely, we have

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{12} = \frac{1}{3}, \quad P(B) = \frac{n(B)}{n(S)} = \frac{6}{12} = \frac{1}{2}, \quad P(C) = \frac{n(C)}{n(S)} = \frac{7}{12},$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{2}{12} = \frac{1}{6}, \quad P(B \cap C) = \frac{n(B \cap C)}{n(S)} = \frac{1}{12},$$

and

$$P(A \cap C) = \frac{n(A \cap C)}{n(S)} = \frac{2}{12} = \frac{1}{6}.$$

Since

$$P(A)P(B) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} = P(A \cap B)$$

the events A and B are independent. Since

$$P(B)P(C) = \frac{1}{2} \cdot \frac{7}{12} \neq \frac{1}{12} = P(B \cap C)$$

the events B and C are dependent. Since

$$P(A)P(C) = \frac{1}{3} \cdot \frac{7}{12} \neq \frac{1}{6} = P(A \cap C)$$

the events A and C are also dependent.

4. (a) (4 points) How many different possible orders are there for the 26 letters in the English alphabet? Justify your answer.
Solution. Since there are 26 letters in the English alphabet, the number of possible orders of the letters is $P(26, 26) = 26!$.

   (b) (4 points) If an order is chosen at random, what is the probability that A is one of the first 5 letters? Justify your answer.
Solution. The probability that A is one of the first 5 letters is the number of orderings where A is one of the first 5 letters divided by the total number of possible orders. The total number of possible orderings where A is one of the first five letters is the product of the number of places in the order that A could appear and the number of possible orderings for the other 25 letters. Specifically, $5 \cdot 25!$, making the probability $\frac{5 \cdot 25!}{26!} = \frac{5}{26}$.

   (c) (4 points) If an order is chosen at random, and A is one of the first five letters, what is the probability that B is one of the first 10 letters? Justify your answer.
Solution. The probability that B is one of the first 10 letters given that A is one of the first five letters is the number of orderings where A is one of the first five letters and B is one of the first ten letters divided by the number of orderings where A is one of the first five letters. The number of orderings where A is one of the first five letters and B is one of the first ten letters is the product of the number of ways to place A and B among the other 24 letters and the number of ways to order the other letters. Choosing one of the first 5 positions for A first and one of the 9 remaining positions in the first 10 B yields that the number of orderings where A is one of the first five letters and B is one of the first ten letters is $5 \cdot 9 \cdot 24!$, making the probability $\frac{5 \cdot 9 \cdot 24!}{5 \cdot 25!} = \frac{9}{25}$.

   (d) (4 points) Are the events "A is one of the first five letters" and "B is one of the first ten letters" independent or dependent events? Explain.
Solution. The probability that B is one of the first 10 letters is the number of orderings where B is one of the first 10 letters divided by the total number of possible orders. By a similar argument to the one used in part (a), $\frac{10 \cdot 25!}{26!} = \frac{10}{26} \neq \frac{9}{25}$. The probability that B is one of the first 10 letters is not equal to the probability that B is one of the first 10 letters given that A is one of the first 5 letters. The

events"A is one of the first five letters" and "B is one of the first ten letters" are dependent. The intuition behind this observation is that if A is in the first 5 positions, then there is one less possible position among the first 10 letters that B could occupy.

5. (16 points) Three cards are drawn without replacement from a standard deck of 52 playing cards.

   Recall that in a standard deck of 52 cards each card has two attributes: a value and a suit. There are four possible suits: hearts, clubs, diamonds and spades. There are thirteen possible values: Ace, 2, ... , 10, Jack, Queen, King.

   (a) How many different possible sets of three cards are there?
   Solution. Since there are 52 cards, we are selecting 3 of the cards, and the order in which we select them does not matter, there are $\binom{52}{3}$ sets of 3 cards.

   (b) What is the probability that all three cards have the same suit?
   Solution. The probability that all three cards have the same suit equals the number of different hands where all 3 cards have the same suit divided by the total number of 3 card hands. The number of 3 card hands where all 3 cards have the same suit is the product of the number of ways to select the suit of the 3 cards and the number of different ways there are to select 3 cards of a given suit. Specifically, there are $4 \cdot \binom{13}{3}$ different hands where all 3 cards have the same suit, and so the probability of getting such a hand is $\frac{4 \cdot \binom{13}{3}}{\binom{52}{3}}$.

   (c) What is the probability that all three cards are Jacks?
   Solution. The probability that all three cards are Jacks equals the number of different hands where all 3 cards are Jacks divided by the total number of 3 card hands. There are $\binom{4}{3}$ ways to choose 3 Jacks so the probability of getting such a hand is $\frac{\binom{4}{3}}{\binom{52}{3}}$.

   (d) Given that exactly two of the three cards are Jacks, what is the probability that the other card is a 10?
   Solution. The probability that the hand contains a 10 given that the hand contains exactly 2 Jacks is the number of hands that contain two Jacks and one 10s divided by the number of hands with two Jacks. The number of hands with two Jacks and one 10 is the product of the number of ways to choose two Jacks and the number of ways to choose one 10, or $\binom{4}{2} \cdot \binom{4}{1}$. The number of ways to choose a hand with exactly two Jacks is the product of the number of ways to

choose two Jacks and the number of ways to choose one non-Jack, or $\binom{4}{2} \cdot \binom{48}{1}$. The probability is $\frac{\binom{4}{2}\cdot\binom{4}{1}}{\binom{4}{2}\cdot\binom{48}{1}} = \frac{4}{48} = \frac{1}{12}$.

6. (15 points) There are two bags on a table in front of you, each with four balls. Bag A has 1 black ball and 3 white balls, Bag B has 2 black balls and two white balls.

   (a) What is the probability that two balls picked from bag A have different colors? Simplify your answer.
   Solution. There are several ways to approach this problem, so we will only give one method related to tree diagrams. Suppose we draw a black ball first, with probability 1/4. Then no matter what happens next, we will end up with 1 black and 1 white. So the probability of drawing B, then W is 1/4. Now, let us get the probability of W, then B. The probability of drawing a white first is 3/4. With that white removed, the probability of drawing the black becomes 1/3. So the probability of drawing W, then B is $3/4 \cdot 1/3 = 1/4$. So the total probability of drawing a white and a black is $1/4 + 1/4 = 1/2$.

   (b) What is the probability that two balls picked from bag B have different colors? Simplify your answer.
   Solution. Using the same reasoning as before, the probability of drawing B then W is $1/2 \cdot 2/3 = 1/3$, and the probability of W then B is $1/2 \cdot 2/3 = 1/3$ for a total probability of drawing a white and a black of $1/3 + 1/3 = 2/3$.

   (c) Suppose you reached into a random bag and picked out two balls at random. If the two balls are different colors, what is the probability that you reached into bag B? Simplify your answer.
   Solution. Let us call the event of drawing two different colored balls D, the event "we reached into bag A" A, and "we reached into bag B" B. This problem is asking us to compute $P(B|D)$, the probability that we reached into bag B given that both balls were different.
   Since we picked a bag at random, $P(A) = 1/2$ and $P(B) = 1/2$. In the first two parts of the problem we computed $P(D|A) = 1/2$ and $P(D|B) = 2/3$. So to compute $P(B|D)$ we can use Bayes' theorem:

$$P(B|D) = \frac{P(B) \cdot P(D|B)}{P(B) \cdot P(D|B) + P(A) \cdot P(D|A)}$$

   Plugging in our numbers from the other parts gives

$$P(B|D) = \frac{1/2 \cdot 2/3}{1/2 \cdot 2/3 + 1/2 \cdot 1/2} = 4/7$$

7. (16 points) Each Sunday a fisherman visits one of three possible locations near his home: he goes to the sea with probability $\frac{1}{2}$, to a river with probability $\frac{1}{4}$, or to a lake with probability $\frac{1}{4}$. If he goes to the sea there is an 80% chance that he will catch fish; corresponding figures for the river and the lake are 40% and 60% respectively.

   (a) Find the probability that, on a given Sunday, he catches fish. Simplify your answer.
   Solution. Let's use the following notation:
   S: he goes to the sea; R: he goes to the river;
   L: he goes to the lake; F: he catches fish.
   The given information is

$$P(S) = \frac{1}{2}, \quad P(F|S) = \frac{4}{5},$$
$$P(R) = \frac{1}{4}, \quad P(F|R) = \frac{2}{5},$$
$$P(L) = \frac{1}{4}, \quad P(F|L) = \frac{3}{5}.$$

   Hence

$$\begin{aligned} P(F) &= P(S)P(F|S) + P(R)P(F|R) + P(L)P(F|L) \\ &= \frac{1}{2} \cdot \frac{4}{5} + \frac{1}{4} \cdot \frac{2}{5} + \frac{1}{4} \cdot \frac{3}{5} \\ &= \frac{13}{20}. \end{aligned}$$

   (b) If, on a particular Sunday, he comes home without catching anything, where is it most likely that he has been? Justify your answer.
   Solution. From (a), $P(F') = 1 - P(F) = \frac{7}{20}$. Hence

$$P(F'|S) = \frac{P(S \cap F'}{P(F')} = \frac{P(S)P(F'|S)}{P(F')} = \frac{\frac{1}{2}(1 - \frac{4}{5})}{\frac{7}{20}} = \frac{2}{7}$$
$$P(F'|R) = \frac{P(R \cap F'}{P(F')} = \frac{P(R)P(F'|R)}{P(F')} = \frac{\frac{1}{4}(1 - \frac{2}{5})}{\frac{7}{20}} = \frac{3}{7}$$
$$P(F'|L) = \frac{P(L \cap F'}{P(F')} = \frac{P(L)P(F'|L)}{P(F')} = \frac{\frac{1}{4}(1 - \frac{3}{5})}{\frac{7}{20}} = \frac{2}{7}.$$

   Since $P(F'|R) > P(F'|S) = P(F'|L)$, it is most likely that he has been to the river.