1. (12 points) Joanne Ha sells silk-screened T-shirts at community festivals and craft fairs. Her marginal cost to produce one T-shirt is \$4. Her total cost to produce 60 T-shirts is \$300, and she sells them for \$9 each.

   (a) Find the linear cost function for Joanne's T-shirt production.
   Solution. $C(x) = mx + b$. We are given that $m = 4$ and $C(60) = 300$. So $C(x) = 4x + b$. We want to find b:

$$\begin{aligned} 300 &= 4 \cdot 60 + b \\ 300 &= 240 + b \\ 60 &= b. \end{aligned}$$

   Therefore, $C(x) = 4x + 60$.

   (b) How many T-shirts must she produce and sell in order to break even?
   Solution. We are given that $R(x) = 9x$, and we want to solve the equation $C(x) = R(x)$ for x:

$$\begin{aligned} 4x + 60 &= 9x \\ 60 &= 5x \\ \frac{60}{5} &= x \\ 12 &= x \end{aligned}$$

   Joanne must produce and sell 12 shirts.

   (c) How many T-shirts must she produce and sell to make a profit of \$500?
   Solution. We want to find x such that $P(x) = 500$, where $P(x) = R(x) - C(x)$.

$$\begin{aligned} 500 &= 9x - (4x + 60) \\ 500 &= 5x - 60 \\ 560 &= 5x \\ \frac{560}{5} &= x \\ 112 &= x \end{aligned}$$

   To make a profit of 500, Joanne must produce and sell 112 shirts.

2. (12 points) For the following problems calculate the specified matrix
(a)

$$\begin{bmatrix} 2 & -1 & 4 & 1 \\ 0 & 3 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 6 & -3 & -3 \\ 4 & 1 & 0 & 0 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 2 & -1 & 4 & 1 \\ 0 & 3 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 6 & -3 & -3 \\ 4 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2+2 & -1+6 & 4+(-3) & 1+(-3) \\ 0+4 & 3+1 & 3+0 & 2+0 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 5 & 1 & -2 \\ 4 & 4 & 3 & 2 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & -3 & 2 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 2 & -4 \\ 1 & 1 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & -3 & 2 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 2 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 6 + (-1) \cdot 2 + 3 \cdot 1 & 1 \cdot 0 + (-1) \cdot (-4) + 3 \cdot 1 \\ 2 \cdot 6 + (-3) \cdot 2 + 2 \cdot 1 & 2 \cdot 0 + (-3) \cdot (-4) + 2 \cdot 1 \end{bmatrix}$$
$$= \begin{bmatrix} 7 & 7 \\ 8 & 14 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 6 & 0 \\ 2 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 2 & -3 & 2 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 6 & 0 \\ 2 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 2 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 6 \cdot 1 + 0 \cdot 2 & 6 \cdot (-1) + 0 \cdot (-3) & 6 \cdot 3 + 0 \cdot 2 \\ 2 \cdot 1 + (-4) \cdot 2 & 2 \cdot (-1) + (-4) \cdot (-3) & 2 \cdot 3 + (-4) \cdot 2 \\ 1 \cdot 1 + 1 \cdot 2 & 1 \cdot (-1) + 1 \cdot (-3) & 1 \cdot 3 + 1 \cdot 2 \end{bmatrix}$$
$$= \begin{vmatrix} 6 & -6 & 18 \\ -6 & 10 & -2 \\ 3 & -4 & 5 \end{vmatrix}$$

3. (12 points) (a) Use an augmented matrix to find the inverse of $A = \begin{bmatrix} 3 & -14 \\ -1 & 5 \end{bmatrix}$.

Solution. Replace the second row $R2$ of the augmented matrix

$$\left[\begin{array}{cc|cc} 3 & -14 & 1 & 0 \\ -1 & 5 & 0 & 1 \end{array}\right]$$

by the sum of the first row ($R1$) and 3 times the 2nd row, that is $3 \cdot R2 + R1 \to R2$ to obtain

$$\left[\begin{array}{cc|cc} 3 & -14 & 1 & 0 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

Multiplying the 2nd row by 14 and adding to the first (i.e. with $R1 + 14 \cdot R2 \to R1$), we obtain

$$\left[\begin{array}{cc|cc} 3 & 0 & 15 & 42 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

Finally, divide the first row with 3 to get

$$\left[\begin{array}{cc|cc} 1 & 0 & 5 & 14 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

Therefore,

$$A^{-1} = \begin{bmatrix} 5 & 14 \\ 1 & 3 \end{bmatrix}$$

(b) Use the inverse of the coefficient matrix from part (a) to solve the system

$$\begin{aligned} 3x - 14y &= 4 \\ -x + 5y &= -1 \end{aligned}$$

Solution. The system is written as

$$\begin{bmatrix} 3 & -14 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Multiplying the above matrix equation from the *left* with $\begin{bmatrix} 5 & 14 \\ 1 & 3 \end{bmatrix}$, which is the inverse of the coefficent matrix, yields

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & 14 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \cdot 4 + 14 \cdot (-1) \\ 1 \cdot 4 + 3 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

Therefore, $x = 6$ and $y = 1$.

4. (12 points) Using the Gauss-Jordan method, solve the system of equations

$$\begin{aligned} x + 1 &= y + z \\ y + 3 &= x + z \\ z + 1 &= x + y \end{aligned}$$

Solution. First, we need to put the system of equations into a standard form with variables on one side and constants on the other. One way to do this (there are others) is:

$$\begin{aligned} -x + y + z &= 1 \\ x - y + z &= 3 \\ x + y - z &= 1 \end{aligned}$$

This corresponds to the augmented matrix

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 3 \\ 1 & 1 & -1 & 1 \end{array}\right)$$

For the first step of Gauss-Jordan, let us multiply the top row by -1 to get

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 3 \\ 1 & 1 & -1 & 1 \end{array}\right)$$

Then subtracting row 1 from each of rows 2 and 3 gives

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 2 & 0 & 2 \end{array}\right)$$

Swapping rows 2 and 3 and dividing each by 2 gives

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Then we just need to add rows 2 and 3 to row 1, giving

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

corresponding to the solution $x = 2, y = 1, z = 2$.

5. (20 points) Suppose you roll two fair dice. Let A represent the event "the sum of the results is even", and let B represent the event "the sum of the results is divisible by 3".

(a) Write out the sample space S and tell whether the outcomes in S are equally likely.
Solution. The sample space S is the set of 36 different outcomes from rolling two dice

$$\begin{array}{rcccccc} S = \{ & 1-1, & 1-2, & 1-3, & 1-4, & 1-5, & 1-6, \\ & 2-1, & 2-2, & 2-3, & 2-4, & 2-5, & 2-6, \\ & 3-1, & 3-2, & 3-3, & 3-4, & 3-5, & 3-6, \\ & 4-1, & 4-2, & 4-3, & 4-4, & 4-5, & 4-6, \\ & 5-1, & 5-2, & 5-3, & 5-4, & 5-5, & 5-6, \\ & 6-1, & 6-2, & 6-3, & 6-4, & 6-5, & 6-6 \quad \} \end{array}$$

(b) Write out A and B and determine whether A and B are mutually exclusive.
Solution. The event A ("the sum is even") is the set

$$\begin{array}{rcccccc} A = \{ & 1-1, & 1-3, & 1-5, & 2-2, & 2-4, & 2-6, \\ & 3-1, & 3-3, & 3-5, & 4-2, & 4-4, & 4-6, \\ & 5-1, & 5-3, & 5-5, & 6-2, & 6-4, & 6-6 \quad \} \end{array}$$

while the event B ("the sum is divisible by 3") is the set

$$\begin{array}{rcccccc} B = \{ & 1-2, & 1-5, & 2-1, & 2-4, & 3-3, & 3-6, \\ & 4-2, & 4-5, & 5-1, & 5-4, & 6-3, & 6-6 \quad \} \end{array}$$

The intersection of A and B is

$$A \cap B = \{1-5, 2-4, 3-3, 4-2, 5-1, 6-6\}$$

which is not empty, so A and B are not mutually exclusive.

(c) Find $P(A)$, $P(B)$, $P(A \cap B)$ and $P(A \cup B)$.
Solution. The probabilities are

$$P(A) = \frac{n(A)}{n(S)} = \frac{18}{36} = \frac{1}{2}$$
$$P(B) = \frac{n(B)}{n(S)} = \frac{12}{36} = \frac{1}{3}$$
$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

To figure out $P(A \cup B)$, we can use the formula for the probability of a union:

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} \\ &= \frac{3 + 2 - 1}{6} \\ &= \frac{2}{3} \end{aligned}$$

6. (12 points) Suppose we are looking at a weather forecast for tomorrow. The weather report gives the probability of rain $P(R) = 40\%$ and the probability of lightning $P(L) = 30\%$. The probability of both rain and lightning is 20%.

(a) What is the probability of rain or lightning? Justify your answer.
Solution. The statement of the problem tells us that $P(R) = 4/10$, $P(L) = 3/10$, and $P(R \cap L) = 2/10$. From this, we can find the probability of rain *or* lightning by

$$P(R \cup L) = P(R) + P(L) - P(R \cap L) = \frac{4}{10} + \frac{3}{10} - \frac{2}{10} = \frac{1}{2}$$

(b) What is the probability of neither rain nor lightning? Justify your answer.
Solution. The event “neither rain nor lightning” is the complement of the event “either rain or lightning” (symbolically: $(R \cup L)' = R' \cap L'$), so the probability is

$$P(R' \cap L') = P((R \cup L)') = 1 - P(R \cup L) = 1 - \frac{1}{2} = \frac{1}{2}$$

(using what we found in part (a)).

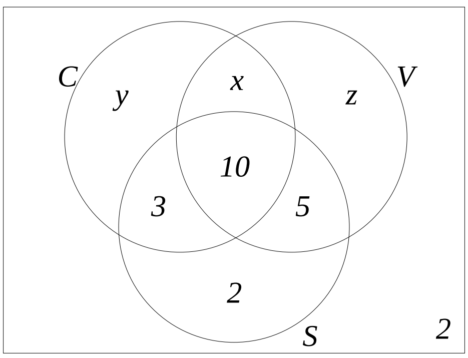
(c) What is the probability of rain but not lightning? Justify your answer.
Solution. The probability of rain but no lightning can be found by taking the probability of rain and subtracting the probability of rain *and* lightning:

$$P(R \cap L') = P(R) - P(R \cap L) = \frac{4}{10} - \frac{2}{10} = \frac{1}{5}$$

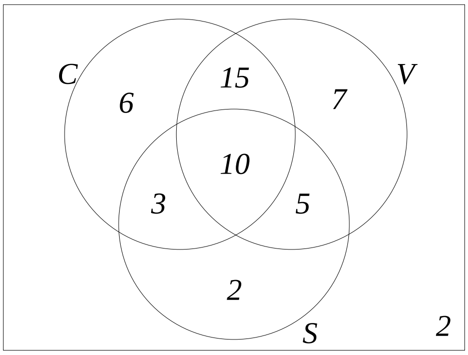
7. (12 points) A survey of 50 children yielded the following data with respect to their ice cream preferences:

10 children like chocolate, vanilla, and strawberry ice cream
13 children like chocolate and strawberry ice cream
15 children like vanilla and strawberry ice cream
34 children like chocolate ice cream
37 children like vanilla ice cream
20 children like strawberry ice cream
2 of the children do not like chocolate, vanilla, nor strawberry ice cream.

(a) Draw a Venn diagram that illustrates the above sets of children, carefully labeling each part of the diagram.
Solution. Let C be the event "the child likes chocolate ice cream", V be the event "the child likes vanilla ice cream", and S be the event "the child likes strawberry ice cream." From the first line of information we know that $n(C \cap V \cap S) = 10$. Further, from the second line of information we can determine that $n((C \cap S) \cap (C \cap V \cap S)') = 3$, and similarly from the third line $n((V \cap S) \cap (C \cap V \cap S)') = 5$. The sixth line of information tells us that $n(S \cap (V \cap S)' \cap (C \cap S)') = 2$. The seventh line tells us that $n((S \cup V \cup C)') = 2$. Putting this information together in a Venn diagram gives us the diagram:



To determine the values of variables x, y, and z we use the information provided in the fourth and fifth lines along with knowing that 50 children participated in the survey. The fourth piece of information tells us that $x + y + 10 + 3 = 34$ and so $y = 21 - x$. The fifth piece of information gives that $x+z+10+5 = 37$, and further that $z = 22-x$. Lastly knowing that the number of children in the survey was 50 tells us that $x+y+z+10+3+5+2+2 = 50$, or by substitution and some algebra that $x = 15$, $y = 6$, and $z = 5$, giving the following updated Venn diagram :



(b) What is the probability that a child selected at random from the 50 surveyed likes chocolate and vanilla ice cream?
Solution. Let U be the sample space for the experiment of choosing a child that participated in the survey at random, and note that the outcomes in U are equally likely and $n(U) = 50$. From the above figure we see that $n(C \cap V) = 10 + 15 = 25$, and $P(C \cap V) = \frac{n(C \cap V)}{n(U)} = \frac{1}{2}$.

8. (8 points) Applying the Gauss-Jordan method to the matrix

$$\left[\begin{array}{cc|c} \frac{-a}{2} & \frac{-b}{2} & \frac{-c}{2} \\ a+d & b+e & c+f \end{array}\right]$$

yields the matrix

$$\left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 4 \end{array}\right]$$

(a) Find the point(s) where the lines $ax + by = c$ and $(a + d)x + (b + e)y = c + f$ intersect. Justify your answer.
Solution. Applying the Gauss-Jordan method to the matrix

$$A = \left[\begin{array}{cc|c} a & b & c \\ a+d & b+e & c+f \end{array}\right]$$

will yield the same matrix as applying the Gauss-Jordan method to

$$B = \left[\begin{array}{cc|c} \frac{-a}{2} & \frac{-b}{2} & \frac{-c}{2} \\ a+d & b+e & c+f \end{array}\right]$$

because performing the row operation $\frac{-R_1}{2} \to R_1$ to A produces B. Since the result of applying the Gauss-Jordan method to A is

$$\left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 4 \end{array}\right]$$

the solution to the system of linear equations $ax + by = c$ and $(a+d)x + (b+e)y = c + f$ is $x = -3$, $y = 4$, and the lines represented by those equations intersect at that point.

(b) Do the lines $ax+by = c$ and $dx+ey = f$ intersect? If so, explain your answer and find the point(s) of intersection. If not, explain your answer.
Solution. Applying the Gauss-Jordan method to the matrix

$$C = \left[\begin{array}{cc|c} a & b & c \\ d & e & f \end{array}\right]$$

will yield the same matrix as applying the Gauss-Jordan method to A because performing the row operation $R_1+R_2 \to R_2$ to C produces A. Since the result of applying the Gauss-Jordan method to C is

$$\left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 4 \end{array}\right]$$

the solution to the system of linear equations $ax + by = c$ and $dx + ey = f$ is $x = -3$, $y = 4$, and the lines represented by those equations intersect at that point.