§6.1 # 28

§6.1: Inverse Functions and Their Derivatives

18. \( f(x) = x^{2/3} \) and \( x \geq 0 \), so \( y = x^{2/3} \Rightarrow y^{3/2} = x \). Thus, \( f^{-1}(x) = x^{3/2} \).

28. \( f(x) = 2x^2 \), \( x \geq 0 \) and \( a = 5 \). \( y = 2x^2 \Rightarrow y^2 = 4x^2 \Rightarrow \sqrt{y^2} = \sqrt{3} = x \), (ONLY since \( x \geq 0 \)) so \( f^{-1}(x) = \sqrt{x} \). Then, \( f'(5) = 4(5) = 20 \) and \( (f^{-1})'(f(5)) = \frac{1}{2\sqrt{2f(5)}} = \frac{1}{2\sqrt{10}} = \frac{\sqrt{2}}{4} \).

30. a) \( h(k(x)) = h((4x)^{1/3}) = \frac{(4x)^{1/3}}{4} = \frac{4x}{4} = x \) and \( k(h(x)) = k(\frac{4x^3}{4}) = (4\frac{x^3}{4})^{1/3} = (x^3)^{1/3} = x \).

b) see graph above

c) \( h'(x) = \frac{3x^2}{4} \), so \( h'(2) = 3 \) and \( h'(-2) = 3 \). Also, \( k'(x) = \frac{4}{3(4x^3)^{2/3}} \), so \( k'(2) = \frac{1}{3} \) and \( k'(-2) = \frac{1}{6} \).

d) \( h'(0) = 0 \), so the line \( y = 0 \) is the tangent line. However, \( \lim_{x \to 0} k'(x) = \infty \), so the line \( x = 0 \) is the tangent line. (Note the limit convention being expressed here: \( \frac{1}{0} = \infty \))

§6.6: L'Hôpital's Rule

8. \( \lim_{t \to 0} \frac{\sin(3t)}{t} \) is indeterminate of the form \( \frac{0}{0} \), so, by L'Hôpital's, this is equal to \( \lim_{t \to 0} \frac{3\cos(3t)}{1} = 5 \).

12. \( \lim_{\theta \to \frac{\pi}{2}} \frac{3\theta + \pi}{\sin(\theta + \frac{\pi}{2})} \) is again of the form \( \frac{0}{0} \), so is equal to \( \lim_{\theta \to \frac{\pi}{2}} \frac{3}{\cos(\theta + \frac{\pi}{2})} = 3 \).

24. \( \lim_{x \to 0} \frac{3^x - 1}{2x - 1} \) is of the \( \frac{0}{0} \) form, so is equal to \( \lim_{x \to 0} \frac{3^x \ln(3)}{2^x \ln(2)} \).

38. \( \lim_{x \to \infty} \frac{1}{x \ln(x)} \int_1^x \ln(t) \, dt \) is of the form \( \frac{\infty}{\infty} \), so is equal to \( \lim_{x \to \infty} \frac{\ln(x)}{\ln(x) + 1} \), which is again of the form \( \frac{\infty}{\infty} \), so is equal to \( \lim_{x \to \infty} \frac{1/x}{1/x} = 1 \).
50. \( \lim_{x \to 0} (e^x + x)^{1/x} \) is indeterminate of the form \( 1^\infty \). We solve this by letting \( y = (e^x + x)^{1/x} \) and looking at \( \lim_{x \to 0} \ln(y) = \lim_{x \to 0} \frac{\ln(e^x + x)}{x} \). This is of the form \( \frac{0}{0} \), so we use l'Hôpital’s rule to find that it equals \( \lim_{x \to 0} \frac{e^x + 1}{e^x + x} = 2 \). Thus, the actual limit we want (that of \( y \), not of \( \ln(y) \)) is \( e^2 \).

54. As the book states, this can’t be done with l’Hôpital’s rule. However, we’ll use the same sort of trick as we did in \# 50; that is, if \( f(x) \) is a continuous function, then \( \lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) \) (we used this fact in \# 50 with \( f(x) = \ln(x) \)). In this case, \( f(x) = \sqrt{x} \). So, \( \lim_{x \to 0^+} \sqrt{\frac{x}{\sin(x)}} = \sqrt{\lim_{x \to 0^+} \frac{x}{\sin(x)}} \). The limit inside is of the form \( \frac{0}{0} \), so we can use l’Hôpital’s again and get \( \sqrt{\lim_{x \to 0^+} \frac{1}{\cos(x)}} = \sqrt{1} = 1 \).