§5.4 # 12

§5.4 # 28

§5.5 # 10

§5.5 # 18

§5.6 # 10

§5.6 # 20
§5.4: Cylindrical Shells
6. For the given graph in the book, we see that \(a = 0\) and \(b = 3\). Thus, \(V = \int_0^3 2\pi (\text{shell radius})(\text{shell height}) \, dx = \int_0^3 2\pi x \left( \frac{9\pi}{\sqrt{x^2+9}} \right) \, dx\). Under a \(u\)-substitution with \(u = x^3 + 9\), this becomes \(V = 2\pi \int_9^{36} 3u^{-1/2} \, du = 6\pi \left[ 2u^{1/2} \right]_{9}^{36} = 12\pi(6 - 3) = 36\pi\).

12. It’s easy to see that \(a = 0, b = 1\). Thus, \(V = \int_0^1 2\pi (\text{shell radius})(\text{shell height}) \, dx = \int_0^1 2\pi x (\sqrt{x} - (2x - 1)) \, dx = 2\pi \int_0^1 x^{3/2} - 2x^2 + x \, dx = 2\pi \left[ \frac{2x^{5/2}}{5} - \frac{2x^3}{3} + \frac{x^2}{2} \right]_0^1 = 2\pi \left( \frac{5}{3} - \frac{2}{3} + \frac{1}{2} \right) = \frac{7\pi}{15}\).

28. (a) Use the washer method: \(V = \int_0^4 \pi [R(x)^2 - r(x)^2] \, dx = \pi \int_0^4 \left( \frac{x^2}{2} + 2 \right)^2 - x^2 \, dx = \pi \int_0^4 -\frac{3}{4} x^2 + 2x + 4 \, dx = \frac{1}{4} \left[ x^2 + x^3 + 4x \right]_0^4 = \pi (-16 + 16 + 16) = 16\pi\).

(b) Use the shell method: \(V = \int_0^4 2\pi (\text{shell radius})(\text{shell height}) \, dx = \int_0^4 2\pi x (\frac{x}{2} + 2 - x) \, dx = 2\pi \int_0^4 2x - \frac{x^2}{2} \, dx = 2\pi \left[ x^2 - \frac{x^3}{6} \right]_0^4 = 2\pi (16 - \frac{64}{6}) = \frac{32\pi}{3}\).

(c) Use the shell method: \(V = \int_0^4 2\pi (\text{shell radius})(\text{shell height}) \, dx = \int_0^4 2\pi (4 - x) (\frac{x}{2} + 2 - x) \, dx = 2\pi \int_0^4 8 - 4x + \frac{x^2}{2} \, dx = 2\pi \left[ 8x - 2x^2 + \frac{x^3}{6} \right]_0^4 = 2\pi (32 - 32 + \frac{64}{6}) = \frac{64\pi}{3}\).

(d) Use the washer method: \(V = \int_0^4 \pi [R(x)^2 - r(x)^2] \, dx = \pi \int_0^4 (8 - x)^2 - (6 - \frac{x}{2})^2 \, dx = \pi \int_0^4 \frac{3x^2}{4} - 10x + 28 \, dx = \pi \left[ \frac{x^3}{4} - 5x^2 + 28x \right]_0^4 = 48\pi\)

§5.5: Lengths of Plane Curves
10. First, \(\frac{dy}{dx} = \frac{3x^{1/2}}{2}\). So, \(L = \int_0^4 \sqrt{1 + \frac{9x}{4}} \, dx\). Using a \(u\)-substitution with \(u = 1 + \frac{9x}{4}\), this becomes \(L = \int_1^{10} \frac{4}{9} u^{1/2} \, du = \left[ \frac{4}{9} \frac{u^{3/2}}{3/2} \right]_1^{10} = \frac{8}{27} (10\sqrt{10} - 1)\).

18. By the Fundamental Theorem of Calculus, \(\frac{dy}{dx} = \sqrt{3x^4 - 1}\). Thus, \(L = \int_{-2}^{2} \sqrt{1 + (3x^4 - 1)} \, dx = \int_{-2}^{2} \sqrt{3x^2} \, dx = \sqrt{3} \left( \frac{x^3}{3} \right)_{-2} = \frac{\sqrt{3}}{3} (-1 + 8) = \frac{7\sqrt{3}}{3}\).

§5.6: Areas of Surfaces of Revolution
10. We’re rotating about the \(y\)-axis, so we solve for \(x\) and find \(\frac{dx}{dy}\): \(x = 2y\), so \(\frac{dx}{dy} = 2\).

Hence, \(S = \int_0^2 2\pi 2y \sqrt{1 + 2^2} \, dy = 4\pi \sqrt{5} \int_0^2 y \, dy = 8\pi \sqrt{5}\). Based on the geometric formula, \(S = 1/2 (\text{base circumference})(\text{slant height}) = 1/2 (8\pi)(\sqrt{4^2 + 2^2}) = 4\pi(2\sqrt{5}) = 8\pi \sqrt{5}\), which agrees with the integral (phew!).

20. Again, we’re rotating about the \(y\)-axis. This time, \(\frac{dx}{dy} = \frac{1}{\sqrt{2y - 1}}\). Thus, \(S = \int_{5/8}^{1} 2\pi \sqrt{2y - 1} \sqrt{1 + \frac{1}{2y - 1}} \, dy = 2\pi \int_{5/8}^{1} \sqrt{(2y - 1) + 1} \, dy = 2\pi \int_{5/8}^{1} \sqrt{2y} \, dy = 2\pi \sqrt{2} \left[ \frac{\sqrt{2y^2}}{3} \right]_{5/8}^{1} = \frac{4\pi\sqrt{2}}{3} \left( 1 - \frac{5\sqrt{5}}{8} \right) = \frac{\pi}{12} (16\sqrt{2} - 5\sqrt{5})\).

28. Using the labels on the picture in the book, we will slice the loaf of radius \(r\) at \(x = a\) and \(x = a + h\), yielding a yummy piece of bread \(h\) units wide. Assuming that
nothing stupid happens (i.e., we assume that $a \geq -r$ and $a + h \leq r$; otherwise you would be cutting where there is no bread), we can find the surface area of our slice using the formulae from this section. Thus, $y = \sqrt{r^2 - x^2}$, so $\frac{dy}{dx} = -\frac{x}{\sqrt{r^2 - x^2}}$.

Hence, $S = \int_a^{a+h} 2\pi \sqrt{r^2 - x^2} \sqrt{1 + \frac{x^2}{r^2 - x^2}} \, dx = 2\pi \int_a^{a+h} \sqrt{(r^2 - x^2) + x^2} \, dx = 2\pi r \int_a^{a+h} \, dx = 2\pi rh$, which doesn’t depend on $a$. Cool!