WHAT IS Outer Space?

Karen Vogtmann

To investigate the properties of a group G, it is often useful to realize G as a group of symmetries of some geometric object. For example, the classical modular group $PSL(2,\mathbb{Z})$ can be thought of as a group of isometries of the upper half-plane $\{(x,y) \in \mathbb{R}^2 | \, y > 0\}$ equipped with the hyperbolic metric $ds^2 = (dx^2 + dy^2)/y^2$. The study of $PSL(2,\mathbb{Z})$ and its subgroups via this action has occupied legions of mathematicians for well over a century.

We are interested here in the (outer) automorphism group of a finitely-generated free group. Although free groups are the simplest and most fundamental class of infinite groups, their automorphism groups are remarkably complex, and many natural questions about them remain unanswered. We will describe a geometric object $\mathcal{O}_n$ known as *Outer space*, which was introduced in [2] to study $Out(F_n)$.

The abelianization map $F_n \to \mathbb{Z}^n$ induces a map $Out(F_n) \to GL(n,\mathbb{Z})$, which is an isomorphism for $n = 2$. Thus the upper half plane used to study $PSL(2,\mathbb{Z})$ can serve equally well for $Out(F_2)$. For $n > 2$ the map to $GL(n,\mathbb{Z})$ is surjective but has a large kernel, so the action of $Out(F_n)$ on the higher-dimensional homogeneous space $SL(n,\mathbb{R})/SO(n,\mathbb{R})$ is not *proper*... stabilizers of points are infinite and difficult to understand. This makes the homogeneous space unsuitable for studying $Out(F_n)$, and something new is needed.

$PSL(2,\mathbb{Z})$ can also be interpreted as the mapping class group of a torus, and the upper half-plane as the Teichmüller space of the torus. This shift in viewpoint motivates the construction we will give here of Outer space. In general, the mapping class group of a closed surface S is the group of isotopy classes of homeomorphisms of S. (Recall that two homeomorphisms are *isotopic* if one can be deformed to the other through a continuous family of homeomorphisms.) One way to describe a point in the Teichmüller space of S is as a pair (X,g), where X is a surface equipped with a metric of constant negative curvature, and $g\colon S \to X$ is a homeomorphism, called the *marking*, which is well-defined up to isotopy. From this point of view, the mapping class group (which can be identified with $Out(\pi_1(S))$) acts on (X,g) by composing the marking with a homeomorphism of S – the hyperbolic metric on X does not change. By deforming the metric on X, on the other hand, we obtain a neighborhood of the point (X,g) in Teichmüller space.

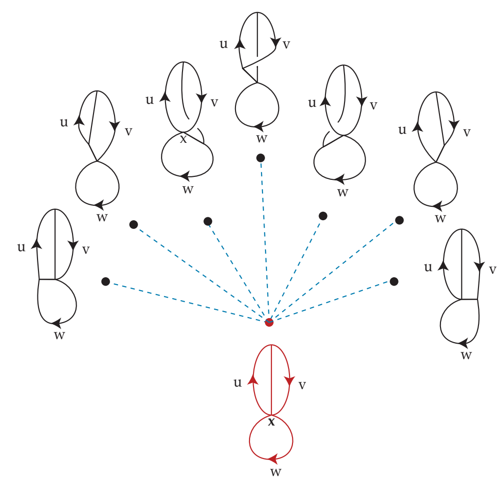

Figure 1: Marked graphs which are close in Outer space

To get a space on which $Out(F_n)$ acts, we imitate the above construction of Teichmüller space, replacing S by a graph R with one vertex and n edges (a *rose with n petals*). However, we no longer insist that the marking g be a homeomorphism; instead we require only that g be a homotopy equivalence, and we take its target X to be a finite graph whose edges are isometric to intervals in $\mathbb{R}$. Thus a point in $\mathcal{O}_n$ is a pair (X,g), where X is a finite metric graph and $g\colon R \to X$ is a homotopy equivalence. Two pairs (X,g) and (X',g') are equivalent if X and X' are isometric and g is homotopic to g' under some isometry. In order to make $\mathcal{O}_n$ finite-dimensional, we also assume that the graphs X are connected and have no vertices of valence one or two. Finally, it is usually convenient to normalize by assuming the sum of the lengths of the edges is equal to one.

The mapping class group acts on Teichmüller space by changing the marking, and the analogous statement is true here: an element of $Out(F_n)$, represented by a homotopy equivalence of R, acts on Outer space by changing only the marking, not the metric graph. A major difference from Teichmüller space appears when one looks closely at a neighborhood of a point. Teichmüller space is a manifold. In Outer space points arbitrarily close to a given point (X, g) may be of the form (Y, h) with Y not homeomorphic to X. An example is shown in Figure 1, where several marked graphs near the red graph are obtained by folding pairs of edges incident to the vertex x together for a small distance. In general there are many different possible foldings, and this translates to the fact that there is no Euclidean coordinate system which describes all nearby points, i.e. Outer space is not a manifold.

Outer space is not too wild, however...it does have the structure of a locally finite cell complex, and it is a theorem that Outer space is contractible. It also has the structure of a union of open simplices, each obtained by varying the edge-lengths of a given marked graph (X, g). For $n = 2$ these simplices can have dimension 1 or 2, but not dimension 0, so Outer space is a union of open triangles identified along open edges (Figure 2).

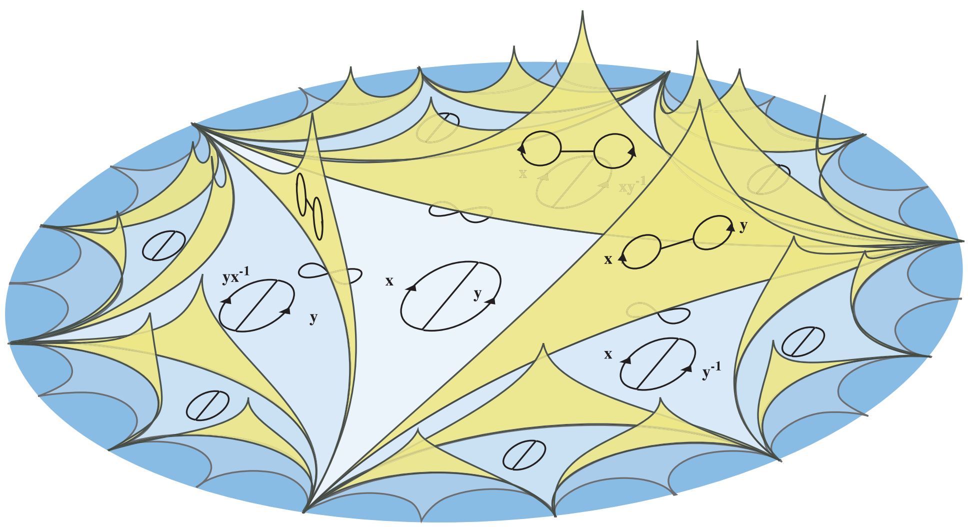

Figure 2: Outer space in rank 2

The stabilizer of a point (X, g) under the action of $Out(F_n)$ is isomorphic to the group of isometries of the graph X. In particular, it is a finite group, so the action is proper. Therefore Outer space serves as an appropriate analog of the homogeneous space used to study a lattice in a semisimple Lie group, or of the Teichmüller space used to study the mapping class group of a surface.

The analogies with lattices and with mapping class groups have turned out to be quite strong. For example, it has been shown that $Out(F_n)$ shares many cohomological properties, basic subgroup structure and many rigidity properties with these classes of groups. The proofs of these facts are frequently inspired by proofs in the analogous settings and use the action of $Out(F_n)$ on Outer space. However, the details are often of a completely different nature and can vary dramatically in difficulty, occasionally being easier for $Out(F_n)$ but more often easier in at least one of the other settings.

Perhaps the most extensive use of Outer space to date has been for computing algebraic invariants of $Out(F_n)$ such as cohomology and Euler characteristic. Appropriate variations, subspaces, quotient spaces and completions of Outer space are also used. For example, the fact that $Out(F_n)$ acts with finite stabilizers on $\mathcal{O}_n$ implies that a finite-index, torsion-free subgroup Γ acts freely. Therefore the cohomology of Γ is equal to the cohomology of the quotient $\mathcal{O}_n/\Gamma$, and vanishes above the dimension of $\mathcal{O}_n$. In fact a stronger statement is true: the cohomology must also vanish in dimensions below the dimension of $\mathcal{O}_n$. In order to find the best bound on this vanishing (called the *virtual cohomological dimension* of $Out(F_n)$), we consider the so-called *spine* of Outer space. This is an invariant subspace K_n of much lower dimension than $\mathcal{O}_n$ which is a deformation retract of the whole space. The dimension of K_n gives a new upper bound for the virtual cohomological dimension, and this upper bound agrees with a lower bound given by the rank of a free abelian subgroup of $Out(F_n)$.

Further uses for the spine of Outer space come from the observation that the quotient $K_n/Out(F_n)$ is compact. As a result one can show, for example, that the cohomology of $Out(F_n)$ is finitely-generated in all dimensions and that there are only finitely many conjugacy classes of finite subgroups. The spine is a simplicial complex on which $Out(F_n)$ acts by permuting the simplices, and in fact it was shown by Bridson and Vogtmann that $Out(F_n)$ is isomorphic to its full group of simplicial automorphisms. Thus the the spine intrinsically contains all possible information about the group!

There are many recent papers on $Out(F_n)$ and Outer space, including work on compactifications, metrics and geodesics, dynamics of the action, embeddings and fibrations, and connections to operads and symplectic representation theory. The interested reader is referred to the three survey articles [1], [3] and [4], for further history and for discussion of some of the recent developments.

References

[1] Martin R. Bridson and Karen Vogtmann. Automorphisms of free groups, surface groups and free abelian groups. In Benson Farb, editor, *Problems on Mapping Class Groups*, number 74 in Proc. Symp. Pure Math., pages 301–316. American Mathematical Society, Providence, RI, 2005.

[2] Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. *Invent. Math.*, 84(1):91–119, 1986.

[3] Karen Vogtmann. Automorphisms of free groups and outer space. *Geom. Dedicata*, 94:1–31, 2002.

[4] Karen Vogtmann. The cohomology of automorphism groups of free groups. In *Proceedings of the International Congress of Mathematicians (Madrid, 2006)*, 2006.