Note Taker Checklist Form -*MSRI*

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Talk Title and Workshop assigned to: Mostow Rigidity / Connections for Women: Geometric Group Theory

Lecturer (Full name): Anna Wienhard

Date & Time of Event: 8/23/07 9-10am & 11-12am

Check List:

(✓) Introduce yourself to the lecturer prior to lecture. Tell them that you will be the note taker, and that you will need to make copies of their own notes, if any.

(✓) Obtain all presentation materials from lecturer (i.e. Power Point files, etc). This can be done either before the lecture is to begin or after the lecture; please make arrangements with the lecturer as to when you can do this.

(✓) Take down all notes from media provided (blackboard, overhead, etc.)

( ) Gather all other lecture materials (i.e. Handouts, etc.)

( ) Scan all materials on PDF scanner in 2nd floor lab (assistance can be provided by Computing Staff) – Scan this sheet first, then materials. In the subject heading, enter the name of the speaker and date of their talk.

Please do **NOT** use **pencil** or colored pens other than black when taking notes as the scanner has a difficult time scanning pencil and other colors.

Please fill in the following after the lecture is done:

1. List 6-12 lecture keywords: strong rigidity, super rigidity, entropy rigidity, locally symmetric spaces, volume entropy, lattices in Lie groups, Patterson-Sullivan measure, quasiisometry, homotopy equivalence

2. Please summarize the lecture in 5 or less sentences.
Lecture presented an algebraic and a geometric form of Mostow's rigidity theorem. A proof of the geometric form was given, based on a theorem by Besson-Courtois-Gallot. Finally, the lecture presented a sketch of the proof of the Besson-Courtois-Gallot theorem.

Once the materials on check list above are gathered, please scan ALL materials and send to the Computing Department. Return this form to Larry Patague, Head of Computing (rm 214)

Mostow Rigidity (1968)

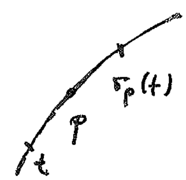
Theorem: (Vague) Let X be a compact connected irreducible locally symmetric space of nonpositive curvature. If $\dim X \geq 3$, then $\pi_1(X)$ determines X uniquely up to isometry and scaling.

Why do we need to exclude $\dim X = 2$?

Then X = Surface of genus $g \geq 2$. Given $\pi_1(X)$ there is a $6g-6$ dimensional moduli space of hyperbolic metrics ($K_X \equiv -1$), which are not isometric.
(Teichmüller space / moduli space)

Let's have a closer look at the Theorem.

(X, g_X) locally symmetric



σ_p is a local isometry.

$(\tilde{X}, \tilde{g}_X)$ symmetric space

σ_p is global isometry
$\{\sigma_p \mid p \in \tilde{X}\}$ generate $\mathrm{Isom}(\tilde{X})$

$G = \mathrm{Isom}(\tilde{X})^\circ$ (connected) (semi)simple Lie group
non-compact ($K_X \leq 0$)

$G \curvearrowright \tilde{X}$ transitively

$\tilde{X} = G/K$ carries a (up to scaling (in factors)) unique G-invariant metric.
this is scaling above

$\pi_1(X) = \Gamma < G$ cocompact lattice.

Algebraic Formulation: Let G, G' (semi)simple, adjoint. connected real Lie groups without centre
with no factors compact or $\cong SL(2,\mathbb{R})$
Let $\Gamma < G$, $\Gamma' < G'$ be cocompact lattices
If $\varphi : \Gamma \to \Gamma'$ is an isomorphism.
Then $\exists\ \bar{\varphi} : G \to G'$ (analytic) isomorphism s.th. $\bar{\varphi}|_\Gamma = \varphi$. $\blacksquare$

In this algebraic formulation Mostow's rigidity result has been generalized by Margulis

<u>Margulis Superrigidity</u> (1973)

Let G be as above with $\mathrm{rk}_{\mathbb{R}} G \geq 2$ (that is $K_X \leq 0$, but X has 0 curvature planes)
and not $K_X < 0$

Let $\Gamma < G$ be an irreducible lattice

Let G' be as above

If $\varphi : \Gamma \to G'$ is a group homomorphism.

Then
- either $\varphi(\Gamma)$ is relatively compact
- or $\exists\, \bar{\varphi} : G \to G'$ s.th. $\bar{\varphi}|_{\Gamma} = \varphi$

• <u>Note</u>: Margulis not true for e.g. $G = SO(1,n)$, but.

<u>Remark</u>. • If $\mathrm{rk}_{\mathbb{R}} G \geq 2$, ~~then~~ Margulis $\Rightarrow$ Mostow.

So, we will focus on X locally symmetric, where G is $\mathrm{rk}_{\mathbb{R}} G = 1$, that is $K_X < 0$:

<u>This means</u>.

$$X = \begin{cases} \mathbb{H}^n_{\mathbb{R}} & n \geq 3 \quad K_X \equiv -1 \\ \mathbb{H}^n_{\mathbb{C}} & n \geq 2 \\ \mathbb{H}^n_{\mathbb{H}} & \\ Ca\mathbb{H}^2 & \end{cases} \quad \Big\} \; \forall \; -4 \leq K_X \leq -1$$

<u>Notation</u>: $n = \dim_{\mathbb{R}} X$

$d =$ dim of base ring

$J_1 \cdots J_{d-1}$ Orthogonal Endom. of $T\tilde{X}$

s.th. $J_k^2 = -Id \quad J_k J_l = J_l J_k \quad \forall\, k \neq l$.

Now let me state the Theorem which we will prove

Theorem (Geometric formulation) $K<0$:

Let (X,g_X), (Y,g_Y) be compact connected locally symmetric spaces with $K_X, K_Y < 0$

$\dim X, \dim Y \geq 3$

If X, Y are homotopy equivalent, that is $\exists\, f : X \to Y$ s.th. $f_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism, then there exists an isometry $F : X \to Y$ such that in the same homotopy class as f, i.e.

$F_* = f_*$ (F is homotopic to f)

We will not follow Mostow's original proof, but we give a proof due to Besson-Courtois-Gallot (90's), ~~Which~~ This only works for $K<0$.

$\boxed{\to \text{challenge: generalize to } K \leq 0.}$

Their idea is that the locally symmetric metric ~~[illegible]~~ is the "best metric" on a locally symmetric manifold, where "best" is measured in terms of a geometric quantity, which is called

<u>Volume Entropy</u>.

(X,g) compact conn. mfd.

$(\tilde{X},\tilde{g})$ universal covering

Define

$$h(X,g) = \lim_{R\to\infty} \frac{1}{R} \log \operatorname{vol}\left(B(x,R)\right) \qquad x \in \tilde{X}$$

(The limit exists and is independent of $x \in \tilde{X}$).

<u>Lemma</u>: If $K_X \leq -a^2 < 0$, then $h(X,g) > 0$

If (as above $h(X,g_X) = n+d-2$.

Theorem (Besson - Courtois - Gallot) (1990's)

Let (X,g) compact conn. mfd. of dim $n \geq 3$

(Y,g_Y) —"— locally symmetric mfd with $K_Y < 0$. ~~with K_Y~~ either $\max K_Y = -1$ or scaling

Let $f: X \to Y$ be a continuous map of non-zero degree

Then

1) $$h(X,g)^n \operatorname{vol}(X,g) \geq h(Y,g_Y)^n \operatorname{vol}(Y,g_Y)\,|\deg f|$$

(that is $h^n(g) = h^n(g_Y)$ and $\operatorname{vol}(X,g) = \operatorname{vol}^n(g_Y)\,|\deg f|$)

2) "=" holds if and only if

f is homotopic to a (locally isometric) Riemannian covering.

(i.e. metric should minimize $h(X,g)$)

Proof of Mostow Rigidity

We have (X,g_X) (Y,g_Y) locally symmetric mfds $K_X, K_Y < 0$ (normalized as above)

$f: X \to Y$ homotopy equivalence, i.e. continuous map of degree 1.

$$\overset{BCG}{\Longrightarrow} \quad h(X,g_X)^n \operatorname{vol}(X,g_X) \geq h(Y,g_Y) \operatorname{vol}(Y,g_Y).$$

Since $f: X \to Y$ is homotopy equivalence $\exists\, g: Y \to X$ with same properties

$$\overset{BCG}{\Longrightarrow} \quad h(Y,g_Y)^n \operatorname{vol}(Y,g_Y) \geq h(X,g_X) \operatorname{vol}(X,g_X),$$

hence we have "="

$\Rightarrow f \simeq F \qquad F: X \to Y$ (local) isometry, (where scaling is 1 by normalization) ∎

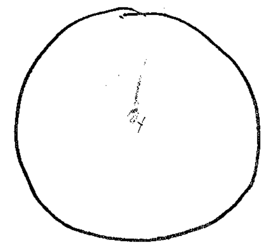
We will now prove BCG-Theorem and in doing so actually construct the map F. $\boxed{\text{Assume also } K_X < 0}$

Try to improve f to make it "better"!

Step 1: A common idea to make a map "better" in rigidity problems is to consider the map $f: X \to Y$ not in the inside, but at infinity.

$$\begin{array}{ccc} \tilde X & \xrightarrow{\tilde f} & \tilde Y \\ \downarrow & & \uparrow \\ \pi(\partial \tilde X) & \xrightarrow{f} & \pi'(\partial \tilde Y) \end{array}$$

Boundary at infinity



(X,g), $K_X < 0$

$(\tilde X, \tilde g)$ universal covering

$\partial \tilde X := \{$ geodesic rays $\xi : \mathbb{R}^+ \to X \} / \sim$

$\xi \sim \xi'$ if $\sup d(\xi(t), \xi'(t)) < \infty$

(topology $T^1_x \tilde X \cong \partial \tilde X$)

$f : X \to Y$ lifts to $f : \tilde X \to \tilde Y$ equivariant $\tilde f(\gamma x) = f_*(\gamma) \tilde f(x)$

Claim: f extends to an equivariant map (homeom) $f : \partial \tilde X \to \partial \tilde Y$

because <u>f is a quasi-isometry</u> (quasi-isometric embedding),

that is. 1) $\exists (K, C)_{K,C>0}$ s.th. $\forall x_1, x_2 \in X$

$$\frac{1}{K} d(x_1,x_2) - C \le d_Y(f(x_1), f(x_2)) \le K\, d(x_1,x_2) + C$$

2) $\exists C'$ s.th. $N_C(f(X)) \supset Y$.

Idea: given geodesic ray $\xi \in \partial \tilde X \to f(\xi)$ is a quasi-geodesic ray

$\xrightarrow[\text{Lemma}]{\text{Morse}} \exists C_{>0} \, \exists ! \eta$ geodesic ray s.th. $f(\xi) \subset N_C(\eta)$

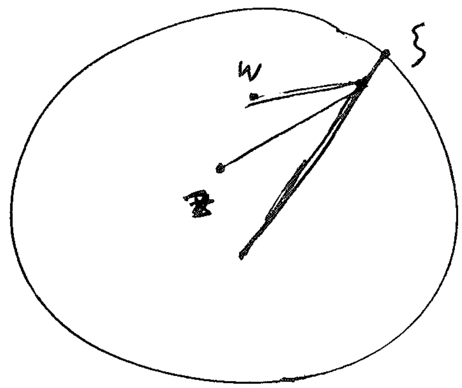
$f(\xi) = \eta \in \partial \tilde Y$.

So from $f : X \to Y$ we got $f : \partial X \to \partial Y$ equivariant.

To use this, we have to understand how we go from $\tilde{X}$ to $\partial \tilde{X}$.

This requires some preparation.

Busemann function



measure distance from infinity

$$B : \partial \tilde{X} \times \tilde{X} \times \tilde{X} \to \mathbb{R}$$

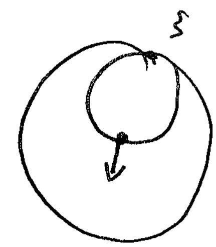
$$B_\xi (z, w) = \lim_{t \to \infty} d(z, \xi(t)) - d(w, \xi(t))$$

we will choose a base pt. $0 \in \tilde{X}$

and consider $B_\xi (z) = B_\xi (z, 0)$

Properties:

- $\to$ C^2 - function
- $\to$ B_ξ = const are horospheres
- $\to$ $\nabla B_\xi^{\tilde{X}}$ is of norm 1, orthogonal to horosphere
- $\to$ $\nabla d B_\xi^{\tilde{X}}$ Hessian is 2nd FF of horosphere.
- $\to$ B_ξ is convex along geodesic rays

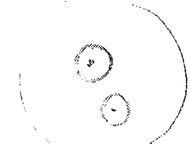


Patterson - Sullivan measures

This is the way to go from $\tilde{X}$ to $\partial \tilde{X}$.

$$\tilde{X} \longmapsto M^1(\partial \tilde{X}) \quad \text{space of probability measures}$$

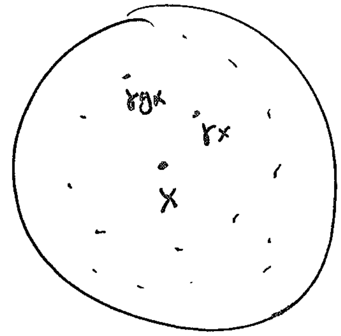
$$x \longmapsto \mu_x$$



such that

- $\to$ $\dfrac{d\mu_x}{d\mu_{x'}}(\xi) = e^{-h(X,g) B_\xi (x, x')}$
- $\to$ $\gamma_* \mu_x = \mu_{\gamma x}$.
- $\to$ no atoms

<u>Construction</u>. for $s > h(X,g)$



$$\mu_x := \lim_{s \to h(X,g)} \boxed{\frac{\displaystyle\sum_{\gamma \in \pi_1(X) = \Gamma} e^{-s\, d(x,\gamma x)}\, \delta_{\gamma x}}{\displaystyle\sum_{\gamma \in \Gamma} e^{-s\, d(x,\gamma x)}}}$$

converges $s > h(X,g)$
diverges for $s \leq h(X,g)$ $\Longrightarrow$ limit is a measure on $\overline{\Gamma x} \cap \partial\tilde{X} = \partial\tilde{X}$.

Γ lattice

for every $s > h(X,g)$ measure on $\tilde{X} \cup \partial\tilde{X}$

With this we substitute $f : \tilde{X} \to \tilde{Y}$ by the following

$$\begin{tikzcd}
\tilde{X} \arrow[r] \arrow[rrr, bend left, "F"] & M^1(\partial\tilde{X}) \arrow[r, "f_*"] & M^1(\partial\tilde{Y}) \arrow[r, dashed, "\text{now we want to get}"] & \tilde{Y} \\
x \arrow[r, mapsto] & \mu_x \arrow[r, mapsto] & f_*\mu_x \arrow[r, mapsto] & \mathrm{bar}(f_*\mu_x)
\end{tikzcd}$$

Natural map

this is done by taking the <u>barycenter</u> of the measure $f_*\mu_x$.

<u>Barycenter</u>

Given $\nu \in M^1(\partial\tilde{Y})$

$$\mathcal{B}(y) = \int_{\partial\tilde{Y}} \mathcal{B}^{\tilde{Y}}_{\eta}(x)\, d\nu(\eta)$$

average distance from infinity

Find the point $z \in \tilde{Y}$ which is closest / equidistant to $\partial\tilde{Y}$ w.r.t ν.

<u>Proposition</u>: Let $\nu \in M^1(\partial\tilde{Y})$ be without atoms

Then $\mathcal{B} : \tilde{Y} \to \mathbb{R}$ is a <u>strictly convex</u> function

Which goes to ∞ if y goes to ∞ along a geodesic.

<u>In particular</u>: $\mathcal{B}$ has a unique critical pt, which is a minimum and is called

$\mathrm{bar}(\nu)$ Barycenter of ν.

Remark: → $\mathrm{bar}(\gamma_* \nu) = \gamma\, \mathrm{bar}(\nu)$

→ $\mathrm{bar}(\mu_y) = y$

Proof: 1) Convexity of $\mathcal{B}$ follows from convexity of $B_\eta^{\tilde{Y}}$

2) For strict convexity consider

$$(\nabla d \mathcal{B})_y (u,u) = \int_{\partial \tilde{Y}} (\nabla d B_\eta)_y (u,u)\, d\nu(\eta)$$

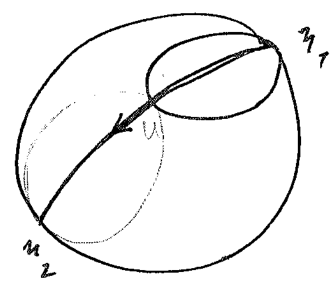
Since $K_Y \leq -a^2 < 0$

we have

$$(\nabla d B_\eta)_y (u,u) \geq \frac{a}{\sqrt{2}}\, g_y(u,u)$$

if $u \in T_y(\text{horosphere})$
$B_\eta \equiv \text{cst}.$

Hence $(\nabla d \mathcal{B})_y (u,u) > 0$ if u is not of the from $\alpha_1 \delta_{\eta_1} + \alpha_2 \delta_{\eta_2}$



3) that $\mathcal{B}(y) \longrightarrow \infty$ for $y \longrightarrow \infty$ is also a consequence of the convexity of $\mathcal{B}^{\tilde{Y}}$, which involves some computation (see BCG-1)

Now that we replaced $f : X \to Y$ by the <u>natural map</u>

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\quad\quad F \quad\quad} & Y \\ \searrow & & \nearrow \\ \mathcal{M}^1(\partial\tilde{X}) & \longrightarrow & \mathcal{M}^1(\partial\tilde{Y}) \end{array}$$

We need to study the properties of F. This will give us a proof of BCG

<u>Proposition</u>: We have for $F: X \to Y$

1) $$|\det dF_x| \leq \frac{h(X,g)^n}{h(Y,g_Y)^n} \qquad \forall x \in X$$

2) if $|\det dF_x| = \dfrac{h(X,g)^n}{h(Y,g_Y)^n}$

then dF_x is a homothety of ratio $\dfrac{h(X,g)}{h(Y,g_Y)}$

<u>Proof of BCG using Proposition</u>

$$\mathrm{vol}(Y,g_Y) = \int_Y 1 = \int_X |\det dF_x| \leq \frac{h(X,g)^n}{h(Y,g_Y)^n}\,\mathrm{vol}(X,g) \quad \Rightarrow 1) \text{ in BCG}$$

for 2)

If we equality ~~then~~ then $\left|\det dF_x\right| = \left(\dfrac{h(X,g)}{h(Y,g_Y)}\right)^n \quad \forall x \in X$ $\cancel{\mathrm{vol}(Y,g_Y) = \mathrm{vol}(X,g)} \cancel{\Leftrightarrow |\det dF_x| = 1 \quad \forall x \in X}$

$\cancel{h(Y,g_Y) = h(X,g)}$

$\overset{2)}{\Rightarrow}$ and hence dF_x is an ~~isometry~~ homothety of ratio $\dfrac{h(X,g)}{h(Y,g_Y)}$

$\boxed{///}$

Now we are left with the technical part:

<u>Proof of Proposition</u>

$F(x)$ is defined by the implicit equation.

$$(dB)_{F(x)}(u) = \int_{\partial\tilde{Y}} \left(dB^{\tilde{Y}}_{\eta}\right)_{F(x)}(u)\, d(f_*\mu_x)(\eta) = 0 \qquad \forall u \in T_{F(x)}\tilde{Y}$$

$$= \int_{\partial\tilde{X}} \left(dB^{\tilde{Y}}_{f(\xi)}\right)_{F(x)}(u)\, d\mu_x(\xi)$$

$$= \int_{\partial\tilde{X}} \left(dB^{\tilde{Y}}_{f(\xi)}\right)_{F(x)}(u)\, e^{-h(X,g)\,B_\xi(x)}\, d\mu_0(\xi).$$

[Write as $G(F(x),x) = 0$ where $G_i(z,x) = \int_{\partial\tilde{X}} \left(dB^{\tilde{Y}}_{f(\xi)}\right)_z (e_i(z))\, d\mu_x(\xi)$. $\quad e_i(z)$ ON-Fam of $T_z\tilde{Y}$

$\Rightarrow F$ is C^1]

If we differentiate we get $\forall v \in T_x \tilde{X}$ that $\forall u \in T_{F(x)} \tilde{Y}$

$$(*)\quad \boxed{\int_{\partial \tilde{X}} (\nabla d B^{\tilde{Y}}_{f(\xi)})_{F(x)} (dF(v), u)\, d\mu_x(\xi) = h(X,g) \int_{\partial \tilde{X}} (dB^{\tilde{Y}}_{f(\xi)})_{F(x)} (u)\, (dB^{\tilde{X}}_{\xi})_x (v)\, d\mu_x(\xi)}$$

Introduce H, K sym. pos. def. endomorphism. of $T_{F(x)} \tilde{Y}$.

$$g_Y(Ku, w) = \int_{\partial \tilde{X}} (\nabla d B^{\tilde{Y}}_{f(\xi)})_{F(x)} (u, w)\, d\mu_x(\xi) \qquad \text{(average of 2nd F.F.)}$$

$$g_Y(Hu, u) = \int_{\partial \tilde{X}} \left((dB^{\tilde{Y}}_{f(\xi)})_{F(x)} (u) \right)^2 d\mu_x(\xi)$$

Then Cauchy-Schwartz-inequality gives (use $(*)$)

$$g_Y(K \cdot dF(v), u)^2 \leq h(X,g)^2 \cdot g_Y(Hu, u) \cdot \int_{\partial \tilde{X}} dB^{\tilde{X}}_{\xi}(v)^2\, d\mu_x(\xi)$$

Lemma: $\displaystyle |\det dF| \leq \frac{h(X,g)^n}{n^{n/2}} \cdot \frac{(\det H)^{1/2}}{\det K}$

Proof: Assume dF is invertible.

let (u_i) be an ONB of $T_{F(x)} \tilde{Y}$

(v_i') basis of $T_x \tilde{X}$ given by $v_i' = (K \circ dF)^{-1}(u_i)$

and (v_i) ONB of $T_x \tilde{X}$ obtained by Gram-Algorithm from v_i'

Then $K \circ dF$ is upper triangular w.r.t $(v_i), (u_i)$, so

$$\det(K \circ dF) = \det K \cdot \det dF = \prod_{i=1}^{n} g_Y(K\, dF(v_i), u_i)$$

So we have

$$(\det K \cdot |\det dF|)^2 \leq h(X,g)^{2n} \cdot \underbrace{\prod_{i=1}^{n} g_Y(Hu_i, u_i)}_{\det H.} \cdot \underbrace{\prod_{i=1}^{n} \int_{\partial \tilde{X}} dB^{\tilde{X}}_{\xi}(v_i)^2 \, d\mu_x(\xi)}$$

$$\leq \left(\frac{\sum \int_{\partial \tilde{X}} dB^{\tilde{X}}_{\xi}(v_i)^2 d\mu_y(\xi)}{n} \right)^n$$

$$\leq \frac{1}{n^n}$$

(because $\sum dB^{\tilde{X}}_{\xi}(v_i) = \|dB^{\tilde{X}}\| =$ and μ_y is probability measure

So we get

$$|\det dF| \leq \frac{h(X,g)^n}{n^{n/2}} \cdot \frac{\det H^{1/2}}{\det K}$$

$\square$

To get further we have to estimate

$$\frac{\det H^{1/2}}{\det K}$$

For this we first write K in terms of H:

Since (Y, g_Y) is locally symmetric of $-4 \leq K_Y \leq -1$

we have (see BCG-1)

$$\nabla dB^{\tilde{Y}}_{\eta}(u,w) = g_Y(u,w) - dB^{\tilde{Y}}_{\eta}(u)\, dB^{\tilde{Y}}_{m}(w) - \sum_{k=1}^{d-1} dB^{\tilde{Y}}_{\eta}(J_k u)\, dB^{\tilde{Y}}_{m}(J_k w)$$

hence

$$\boxed{K = \mathrm{Id} - H - \sum_{k=1}^{d-1} J_k H J_k}$$

Moreover

$$\operatorname{tr} H = \sum g_Y(Hu_i, u_i) = \int_{\partial \tilde{Y}} \sum_{i=1}^{n} (dB^{\tilde{Y}}_{m})_{F(x)}(u_i)^2 \, d(F_*\mu_x)(\xi) = 1$$

because $\|dB^{\tilde{Y}}\| = 1$ for μ_x probability

<u>Lemma</u> (BCG 1) let H be pos def. symmetric, $\mathrm{tr}\, H = 1$, $n \geq 3$. Then

1) $$\frac{\det H^{1/2}}{\det K} \leq \frac{n^{n/2}}{(n+d-2)^n} = \frac{n^{n/2}}{h(Y,g_Y)^n}$$

2) "=" occurs if and only if $H = \frac{1}{n}\mathrm{Id}$.

<u>Assume this lemma</u>

Then $$|\det dF| \leq \frac{h(X,g)^n}{n^{n/2}} \cdot \frac{\det H^{1/2}}{\det K} \leq \frac{h(X,g)}{h(Y,g_Y)^n} \quad 1)\checkmark \quad \square$$

for 2) if we have equality, then $H = \frac{1}{n}\mathrm{Id}$

and $K = \frac{n+d-2}{n} I$

Cauchy Schwarz
So $(*)$ gives

$$g_Y(K \circ dF(v), u)^2 = h(X,g)^2 \, g_Y(Hu,u) \cdot \int_{\partial\tilde{X}} (dB_\xi^{\tilde{X}})(v)^2 \, d\mu_x(\xi)$$

$$g_Y(dF(v), u)^2 = \left(\frac{h(X,g)}{h(Y,g_Y)}\right)^2 \cdot g_Y(u,u) \cdot n \int_{\partial\tilde{X}} (dB_\xi^{\tilde{X}})_x^2(v) \, d\mu_x(\xi)$$

Set $u = dF(w)$.

then $$\|dF(v)\|_x^2 \doteq \left(\frac{h(X,g)}{h(Y,g_Y)}\right)^2 \cdot n \cdot \underbrace{\int_{\partial\tilde{X}} (dB_\xi^{\tilde{X}})_x^2(v) \, d\mu_x(\xi)}_{\leq 1}$$

Set $L = dF^* dF : T_x\tilde{X} \circlearrowleft$

Then $$\det L = |\det dF|^2 = \left(\frac{h(X,g)}{h(Y,g_Y)}\right)^{2n}$$

$$\mathrm{tr}\, L = \sum_{i=1}^n g_x(Lv_i, v_i) = \sum_{i=1}^n g_Y(dF(v_i), dF(v_i)) \leq n\left(\frac{h(X,g)}{h(Y,g_Y)}\right)^2$$

$$\Rightarrow \det(L) = \left(\frac{\mathrm{tr}\, L}{n}\right)^n \Rightarrow L = \frac{\mathrm{tr}\, L}{n}\mathrm{Id} = \left(\frac{h(X,g)}{h(Y,g_Y)}\right)^2 \mathrm{Id}$$, so $g_Y(dF(v_i), dF(v_i)) = \frac{h(X,g)^2}{h(Y,g_Y)^2} g_X(v_i, v_i)$

$\Rightarrow dF$ is an isometry composed with a scaling by $\frac{h(X,g)}{h(Y,g_Y)}$. $\square$

History and Impact of Mostow rigidity Theorem

[Good reading: R. Spatzier, An invitation to rigidity theory.
in: Modern dynamical systems and applications, 211–231.]

The strong rigidity thm of Mostow started with local rigidity proven by ~~Selberg~~ Selberg '1960
then gen. by
Calabi, Vesentini, Weil.

Local rigidity: Let G be a semisimple Lie group
no compact factor or $SL(2,\mathbb{R})$, $SL(2,\mathbb{C})$ factor.
Let $\Gamma < G$ cocompact lattice
if ρ_t is a continuous path of embeddings $\Gamma \hookrightarrow G$
with $\rho_0 = id$, then ρ_t is conjugate to ρ_0.

One important successor to Mostow rigidity thm in its algebraic formulation was Margulis superrigidity for $rk_{\mathbb{R}} G \geq 2$, which implies that every lattice is arithmetic.

This holds also for $\mathbb{H}^n_{\mathbb{H}}$ ($n \geq 2$) and $\mathrm{Ca}\mathbb{H}$, but not for $\mathbb{H}^n_{\mathbb{R}}$ or $\mathbb{H}^n_{\mathbb{C}}$
↳ arithmeticity $n \geq 3$ open question

In its geometric formulation one important generalization of MR is

Borel conjecture: M, N aspherical mfds, if $M \underset{h.e}{\simeq} N$ then $M \underset{homeo}{\simeq} N$

(Farrell-Jones for nonpositively curved mfds).

One important step in our proof and also in Mostow's original prove was that we can assume that $f: \tilde{X} \to \tilde{Y}$ is a quasi-isometry.

This open a new and very active current research area in geometric group theory:

Quasi-isometric rigidity:
→ find invariants of quasi-isometry type. ?
→ When is a q.i at bdd. distance of an isometry? ← Mostow shows this for equivariant ones
→ classify things up to q.i

If time permits (or put on page 13/14 no original proof)

Why is f is a quasi isometry? Uses $\boxed{\text{Compactness of } X, Y}$

Since X, Y compact can assume

1) $f : X \to Y$ (is homotopic to) differentiable map, same for $g : Y \to X$

2) df, dg are uniformly bdd by K

i.e. $d_Y(\tilde{f}(x_1), f(x_2)) \leq K\, d_X(x_1, x_2)$

$d_X(g(y_1), g(y_2)) \leq K \cdot d_Y(y_1, y_2)$.

X compact $\Rightarrow \Gamma \curvearrowright \tilde{X}$ has compact fundamental domain

Since $(g \circ f)_* = id_{\pi_1(X)}$ $\quad g \circ f : \tilde{X} \circlearrowleft$ commutes with every $\gamma \in \Gamma$

$\Rightarrow \exists\, b$ s.th. $d_X(x, g \circ f(x)) \leq b$

Hence

$$d_X(gf(x_1), gf(x_2)) \geq d(x_1, x_2) - 2b$$

and we get

$$d_Y(f(x_1), f(x_2)) \geq \frac{1}{K} d_X(gf(x_1), gf(x_2)) \geq \frac{1}{K} d_X(x_1, x_2) - \frac{2b}{K}. \quad \square$$

Mostow's original approach

1) $f : \tilde{X} \to \tilde{Y}$ quasi-isometry
$\Rightarrow \varphi : \partial\tilde{X} \to \partial\tilde{Y}$ quasi-conformal

2) φ is conformal (ergodicity of Γ-action on $\partial\tilde{X}, \partial\tilde{Y}$)
+ invariance of distortion of φ

3) φ extends to a Γ-equiv. isometry
$F : \tilde{X} \to \tilde{Y}$.

Lit.: BCG: Minimal Entropy and Mostow's rigidity thm, Erg. Th. Dyn. Syst. 16 (1996), no. 4, 623–649

BCG Entropies et rigidités des espaces localement symétriques de courbure strictement négative

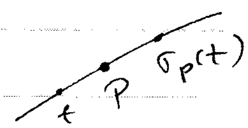
GAFA 5 (1995), no. 5, 731–799

Spatzier.

8/23 Anna Wienhard : Mostow rigidity I

Mostow's (Strong) Rigidity Theorem

(Vague Form) If X connected compact locally symm mfd $K_X \leq 0$ $\dim X \geq 3$, irreducible. Then $\pi_1(X)$ determines X uniquely up to isometry and scaling.
(1968)

(X, g_X) loc. symmetric 

geodesic refl σ_p is a local isometry
$(\tilde{X}, \tilde{g}_X)$ σ_p global isometry
symm space $\{\sigma_p \mid p \in \tilde{X}\}$ generates $\mathrm{Isom}(\tilde{X})$
let G the conn. component of the identity
$G = \mathrm{Isom}(\tilde{X})^0 \curvearrowright \tilde{X}$
$\tilde{X} = G/K$ $\pi_1(X) = \Gamma < G$ cocompact lattice

(Algebraic formulation)
let G, G' connected semisimple real Lie Groups without center, without compact ~~factors~~ factors, without factors $\simeq SL(2,\mathbb{R})$

$$\begin{matrix} G & G' \\ \vee & \vee \\ \Gamma & \Gamma' \end{matrix}$$ cocompact lattices.

If $\varphi : \Gamma \to \Gamma'$ is an isomorphism, then $\exists \bar{\varphi} : G \to G'$ isomorphism s.t. $\bar{\varphi}|_\Gamma = \varphi$

Margulis Superrigidity Theorem (1973)

let G, G' as above, assume $rk_{\mathbb{R}} G \geq 2$

(this means that $K_{\tilde{X}} \leq 0$ but <u>not</u> $K_{\tilde{X}} < 0$)

let $\Gamma < G$ discr. subgr (irreducible)

let $\varphi : \Gamma \to G'$ be a homomorphism.

Then either $\varphi(\Gamma)$ is rel. compact or $\exists\, \bar{\varphi} : G \to G'$ homomorphism s.t. $\bar{\varphi}|_{\Gamma} = \varphi$.

We will assume that our loc. symmetric mfds X, Y are with $K_X < 0, K_Y < 0$.

<u>This means</u>: $\tilde{X} = \mathbb{H}^m_{\mathbb{R}}$ $\quad K_X \equiv -1$

or $\tilde{X} = \mathbb{H}^m_{\mathbb{C}}$ $\quad -4 \leq K_X \leq -1$

or $\tilde{X} = \mathbb{H}^m_{\mathbb{H}}$

or $Ca\mathbb{H}$

$n = \dim_{\mathbb{R}} X$

d = dim of base alg $(1, 2, 4, 8)$

$T_x\tilde{X}$ $\quad J_1, \ldots, J_{d-1}$ endomorphism

$J_k^2 = -id$

$J_k J_\ell = -J_\ell J_k$ if $k \neq \ell$.

Geometric Formulation of Mostow's Th

Let (X, g_x) (Y, g_y) loc. symm. mfds, with $K_X, K_Y < 0$ (normalized as above)
dim X, dim $Y \geq 3$.
If $X \cong Y$ (homotopy equiv), i.e $\exists$ $f: X \to Y$ cont and
$f_*: \pi_1(X) \to \pi_1(Y)$ is isomorphism.
Then $\exists$ $F: X \to Y$ in the same homotopy class as f which is an isometry.

Idea: the locally symm metric on X should be the "best" metric.

<u>Volume entropy</u>.
(X, g) compact connected manifold
$(\tilde{X}, \tilde{g})$ univ. covering

$h(X, g) := \lim_{R \to \infty} \frac{1}{R} \log \mathrm{vol}(B(x, R))$, $x \in \tilde{X}$, limit exists and does not depend on x.

Theorem (Besson - Courtois - Gallot) 1990's
(Y, g_Y), g_Y the metric making Y loc. symm. space.
$K_Y < 0$, dim $Y \geq 3$ (normalized $-4 \leq K_Y \leq -1$)
connected, compact.
(X, g) compact connected manifold, $K_X < 0$
Given $f: X \to Y$ continuous, of non-zero degree,
then 1) $h(X, g)^n \cdot \mathrm{vol}(X, g) \geq h(Y, g_Y)^n \cdot \mathrm{vol}(Y, g_Y) |\deg f|$
2) Equality in "1)" occurs iff f is homotopic to $F: X \to Y$ Riemannian covering.
F is an isometry composed with a scaling $\frac{h(X, g)}{h(Y, g_Y)}$

Mostow by using BGC.

Proof: We have $f: (X, g_X) \to (Y, g_Y)$ cont, $\deg f = 1$

hence (BCG) $\Rightarrow h(X, g_X)^n \operatorname{vol}(X, g_X) \geq h(Y, g_Y)^n \operatorname{vol}(Y, g_Y)$

Similarly we have the inverse map of f,

$$g : (Y, g_Y) \to (X, g_X)$$

(BCG) $\Rightarrow h(Y, g_Y)^n \operatorname{vol}(Y, g_Y) \geq h(X, g_X)^n \operatorname{vol}(X, g_X)$

So we have equality.

By part 2) of BCG f is homotopic to $F : X \to Y$ is a covering of deg 1, hence an isometry.

Part II Idea on proving BCG.

Replace $f : X \to Y$ by a map $\tilde{X} \xrightarrow{f} \tilde{Y}$ with

$$f(\gamma x) = f_*(\gamma) f(x) \quad , \forall x \in \tilde{X}$$

Replace $\tilde{X} \xrightarrow{f} \tilde{Y}$ by F

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f} & \tilde{Y} \\ ③ \downarrow & \overset{F}{\searrow\nearrow} & \uparrow ④ \\ M'(\partial\tilde{X}) & \xrightarrow{②} & M'(\partial\tilde{Y}) \\ ① & & \end{array}$$

① **Boundary at Infinity**

(X, g) comp conn manifold, $K_X < 0$

take the univ. cover $(\tilde{X}, \tilde{g})$.

$$\partial\tilde{X} := \{\xi : \underset{\text{geodesic ray}}{\mathbb{R}^+ \to \tilde{X}}\} / \sim \qquad \text{Space of equiv. class. of geodesic rays}$$

$$\xi \sim \xi' \quad \text{if} \quad \sup_{t \in \mathbb{R}^+} d(\xi(t), \xi'(t)) < \infty$$

Identify $\partial\tilde{X} \simeq T^1_{x_0}(X)$ → which gives the topology.

Claim: $f : \tilde{X} \to \tilde{Y}$ is a quasi-isometry, i.e. $\exists k, c > 0$ s.t.

$$\frac{1}{k} d_X(x_1,x_2) - c \le d_Y(f(x_1), f(x_2)) \le k\, d_X(x_1,x_2) + c$$

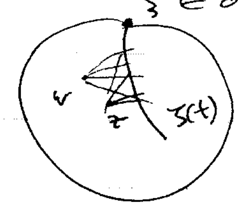
Proof of claim uses that X, Y are compact.

So given $\xi(t) \subset X$ geodesic ray, then $f(\xi(\cdot)) : \mathbb{R}^+ \to \tilde{Y}$ is a quasi geodesic ray.

Morse lemma: Every quasi-geodesic ray is contained in the nhood of a unique geodesic ray η
($K_Y < 0$)

So we get $f : \partial\tilde{X} \to \partial\tilde{Y}$ s.t. $\xi \longmapsto \eta =: f(\xi)$ continuous which gives us (2) by pushing forward the prob. measures through f.

$\xi \in \partial\tilde{X}$



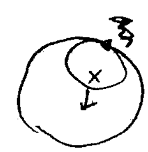
(3) Busemann function

$$B : \partial\tilde{X} \times \tilde{X} \times \tilde{X} \to \mathbb{R}$$

$$B_\xi(z, w) = \lim_{t \to \infty} d(z, \xi(t)) - d(w, \xi(t))$$

Pick $o \in \tilde{X}$ $\quad B_\xi(z) := B_\xi(z, o)$

Properties
a) B is a C^2-function
b) $\{B_\xi(\cdot) \equiv \text{cnst}\}$ is a horosphere



c) $B_\xi(\cdot)$ is convex (along geodesic rays)
d) $(\nabla B_\xi)_z$ is unit normal vector to the horosphere
e) $(\nabla d B_\xi)_z$ Hessian is 2 fundamental form of horosphere

Patterson – Sullivan measures

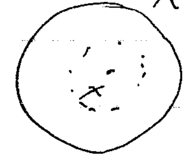
We define

$$\tilde X \longrightarrow M^1(\partial \tilde X)$$
$$x \longrightarrow \mu_x \quad \text{s.t.: Properties} \rightarrow \frac{d\mu_x}{d\mu_{x'}}(\xi) = e^{-h(X,g) B_\xi(x,x')}$$

$\rightarrow \mu_{\gamma_* x} = \gamma_* \mu_x$

$\rightarrow \mu_x$ has no atoms.

$\tilde X$



So how do we construct μ_x?

$\Gamma = \pi_1(X)$

Take the orbit γx, let $s > h(X,g)$

then $$\frac{\sum_{\gamma\in\Gamma} e^{-s\, d(x,\gamma x)} \delta_{\gamma x}}{\sum_{\gamma \in \Gamma} e^{-s\, d(x,\gamma x)}}$$ is a measure on $\tilde X \cup \partial \tilde X$.

(where denominator converges if $s > h(X,g)$ diverges otherwise)

Let $$\mu_x := \lim_{s \to h(X,g)} \frac{\sum_{\gamma\in\Gamma} e^{-s\, d(x,\gamma x)} \delta_{\gamma x}}{\sum_{\gamma\in\Gamma} e^{-s\, d(x,\gamma x)}}$$

then $\mu_x \in M^1(\partial \tilde X)$

Then in (2) $\quad \mu_x \longmapsto f_* \mu_x$.

(4) To go from $M^1(\partial \tilde Y) \longrightarrow \tilde Y$

$$\nu \longmapsto \cancel{\text{bar}}(\nu) := z$$

ν without atoms, z should minimise ~~the~~ average dist. from $\partial \tilde Y$ w.r.t. ν.

Define $$\mathcal{B}(z) := \int_{\partial \tilde Y} B_\xi(z)\, d\nu(\xi)$$

Proposition: If ν has no atoms, then $B : \tilde{Y} \to \mathbb{R}$ is a strictly convex function and $B \to \infty$ if $z \to \infty$ along a geodesic ray.

$\Rightarrow$ B has a unique minimum on $\tilde{Y}$, this is called $\cancel{bar}(\nu)$

We have a natural map $F: \tilde{X} \longrightarrow M'(\partial \tilde{X}) \longrightarrow M'(\partial \tilde{Y}) \xrightarrow{bar} \tilde{Y}$

$$x \longmapsto \mu_x \longmapsto f_* \mu_x \longmapsto bar(f_* \mu_x)$$

equivariant.

Proposition: F is C^1 and 1) $|\det dF_x| \leq \left(\dfrac{h(X,g)}{h(Y,g_Y)} \right)^n$

2) equality in 1) iff dF_x is a homothety by $\dfrac{h(X,g)}{h(Y,g_Y)}$

Proof of Theorem BCG: $vol(Y, g_Y) = \int_Y 1 = \int_X |\det dF_x| \leq \dfrac{h(X,g)^n}{h(\cancel{Y}, g_Y)^n} \cdot vol(X,g)$

which is part 1) of BCG.

If equality $\Rightarrow$ $|\det dF_x| = \dfrac{h(X,g)^n}{h(Y,g_Y)^n}$ $\forall x \in X$

which goes into part 2) of proposition.

Borel Conjecture: Let X, Y aspherical manifolds

If $X \simeq Y$ (homotopy equiv) $\Rightarrow$ $X \simeq Y$ homeomorphic.