Note Taker Checklist Form -*MSRI*

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Talk Title and Workshop assigned to:
Thurston's classification of surface automorphisms/
Connections for women: Geometric Group Theory

Lecturer (Full name): Genevieve Walsh

Date & Time of Event: 8/23/07 1:30-2:30 & 4-5.

Check List:

( ) Introduce yourself to the lecturer prior to lecture. Tell them that you will be the note taker, and that you will need to make copies of their own notes, if any.

( ) Obtain all presentation materials from lecturer (i.e. Power Point files, etc). This can be done either before the lecture is to begin or after the lecture; please make arrangements with the lecturer as to when you can do this.

( ) Take down all notes from media provided (blackboard, overhead, etc.)

( ) Gather all other lecture materials (i.e. Handouts, etc.)

( ) Scan all materials on PDF scanner in 2nd floor lab (assistance can be provided by Computing Staff) – Scan this sheet first, then materials. In the subject heading, enter the name of the speaker and date of their talk.

Please do **NOT** use **pencil** or colored pens other than black when taking notes as the scanner has a difficult time scanning pencil and other colors.

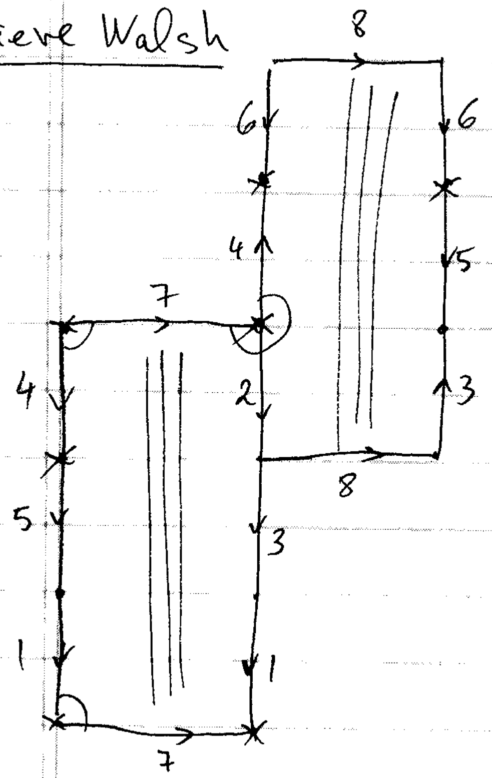
Please fill in the following after the lecture is done:

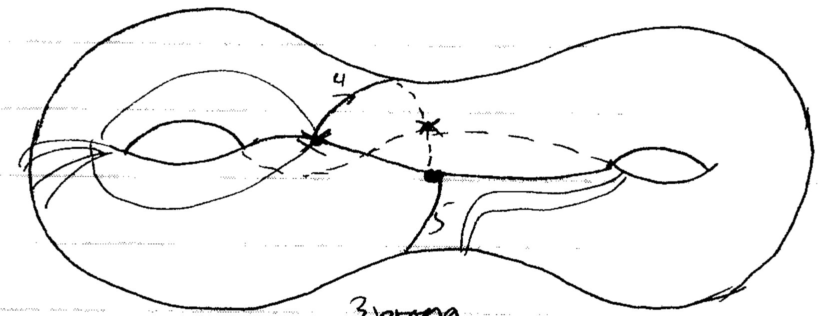
1. List 6-12 lecture keywords: Classification of Surface diffeomorphisms, Teichmüller Space, Thurston, Mapping Class group

2. Please summarize the lecture in 5 or less sentences.
lecture presented the classification of surface homeomorphisms for orientable connected surfaces ~~of~~ with genus $g \geq 1$.

Once the materials on check list above are gathered, please scan ALL materials and send to the Computing Department. Return this form to Larry Patague, Head of Computing (rm 214)

Genevieve Walsh





3prong singularities

Classification of surface homeomorphisms

→ classifying elements of Mod(S) ; S = surface, orientable genus $g \geq 1$ (usually $g \geq 2$) no bdry, connected

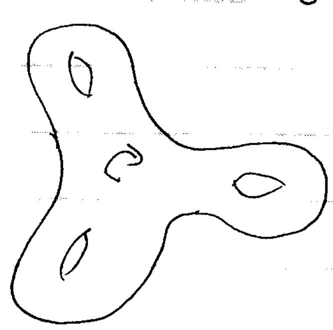
→ $\mathrm{Homeo}^+(S)$ = orientation preserving homeo

→ $\mathrm{Homeo}_0(S)$ = those orient. pres. homeo which are isotopic to the identity

Definition. $\mathrm{Mod}(S) := \dfrac{\mathrm{Homeo}^+(S)}{\mathrm{Homeo}_0(S)}$

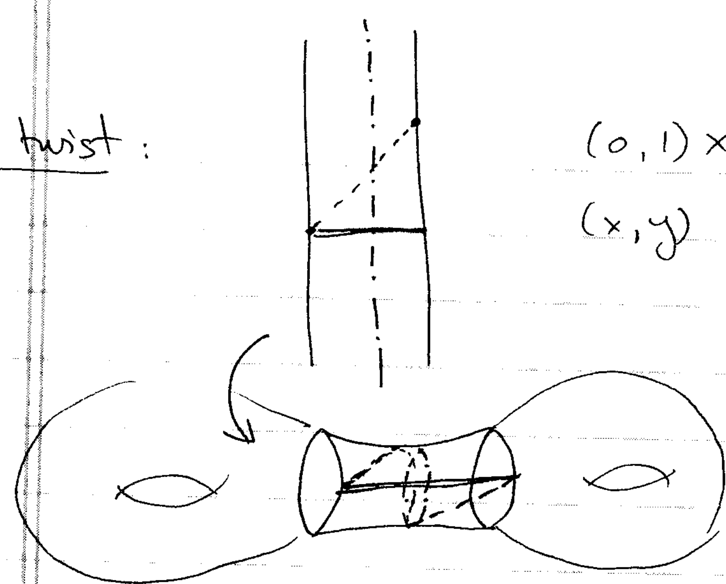
Examples:

(1)



periodic elements of Mod(S) (in picture have per. 3)

2) Dehn twist:



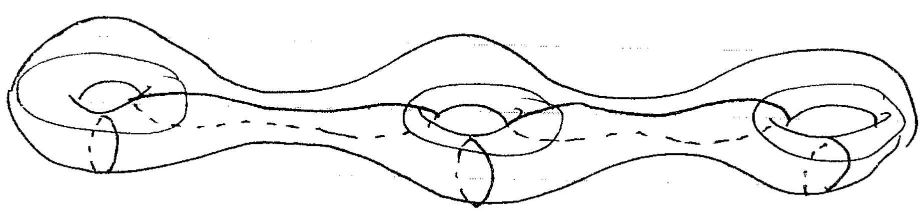
$(0,1) \times \mathbb{R}$

$(x,y) \longmapsto (x, y + 2\pi x)$

core of annulus $= \alpha$

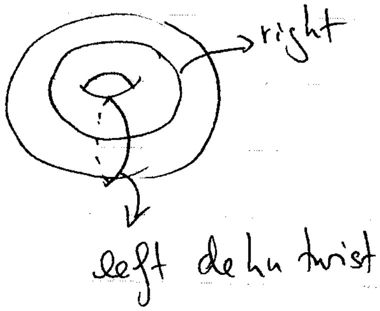
T_α

Dehn (1922) Mod(S) is generated by (finitely many) Dehn twists. $3g-1$ generators



Teruaki www.math.meiji.ac.jp/~ahora/teruaki.html

3



$T_b T_a^{-1}$ $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$!

two eigenvectors $\left(\frac{1 \pm \sqrt{5}}{2}, 1\right)$

eigenvalues $\frac{3 \pm \sqrt{5}}{2}$

2 irrational directions left invariant by this map. in the cover $\rightarrow$ projects to two irrational foliations left invariant. In the case of the torus they are called Anosov.

Teichmüller space S_g, $g \geq 2$

S $\quad \chi(S) < 0$

Euler characteristic

X surface w/ hyperbolic metric

$f : S \to X$ homeomorphism

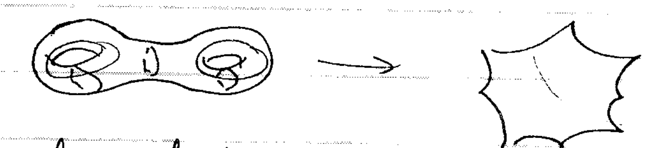
$\mathrm{Teich}(S) = (X, f)/\sim$ where

$(X_1, f_1) \sim (X_2, f_2)$ if $\exists$ isometry s.t. diagram commutes up to homotopy

$$\begin{tikzcd} S \arrow[r, "f_1"] \arrow[dr, "f_2"'] & X_1 \arrow[d, "\text{isom}"] \\ & X_2 \end{tikzcd}$$

i.e. $f_2 \circ f_1^{-1} \sim$ isometry.

$\ell_\gamma : \mathrm{Teich} \to \mathbb{R}$

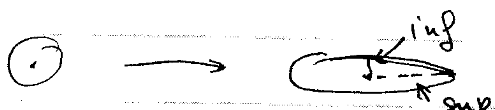


$\ell(X, f) =$ lenght of geod isot. to γ

$h : X \to Y$ homeomorphism

dilatation of h at p is

$$K_p(h) = \lim_{r \to 0} \frac{\sup_{q \in S_p'(r)} d(h(p), h(q))}{\inf_{q \in S_p'(r)} d(h(p), h(q))}$$



tells how far you are from being conformal

let $K(h) = \sup_{p \in X} K_p(h)$. If finite, h is quasi-conformal

Teich distance (X, f) (Y, g)

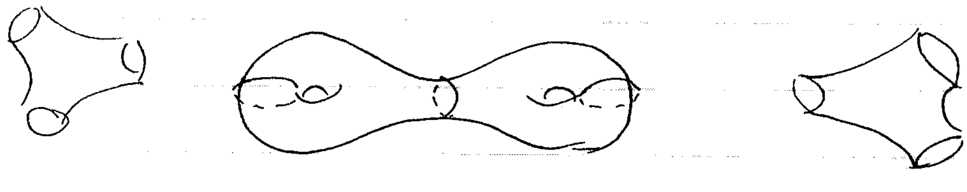
$$d_{\mathrm{Teich}}((X,f),(Y,g)) = \inf\{\log K(h) : h \sim g \circ f^{-1} \text{ is a quasiconf homeo}\}$$

$$\begin{tikzcd} & X \arrow[d, "h \sim g\circ f^{-1}"] \\ S \arrow[ur, "f"] \arrow[r] & Y \end{tikzcd}$$

For $g=1$, the Teich space is the space of marked unit area lattices.

$\text{Teich}(S) \cong \mathbb{R}^{6g-6}$

Divide every surf into pairs of pants



hyperb structures on pairs of pants are determ by lengths.

The coordinates are $(\ell_1, \dots, \ell_{3g-3}, \theta_1, \dots, \theta_{3g-3})$

twists on gluing.

Wolpert area form: $\omega = \sum d\ell_i \wedge d\theta_i$ does not depend on choice

<u>Back to Mod(S)</u>. Mod(S) acts on Teich space by changing the marking:

$$h(X,f) = (X, f\circ h^{-1}) \qquad \overset{h^{-1}}{\circlearrowleft} S \xrightarrow{f} X$$

mapping class

$$\Rightarrow M(S) = \text{Teich}(S) / \text{Mod}(S)$$

Facts about this action: • <u>Action is by isometries</u>

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "f'"'] & X \arrow[d, "h"] \\ & X' \end{tikzcd}$$

h is K quasi conformal

$h\circ f = f'$ in the right homotopy class

$g \in \mathrm{Mod}(S)$ $\quad h \circ (f \circ g^{-1}) \sim f' \circ g^{-1}$

minimal dilatation map has the same K.

- Action is properly discontinuous

Γ is acting on a space; K compact subsp of sp.
$\{\gamma \in \Gamma : \gamma K \cap K \neq 0\}$ is finite $\Rightarrow$

M(S) is an orbifold.

$M_\varepsilon(S)$ = the ε-thick part of moduli space
$$\{X \in M(S) \mid \ell(X) \geq \varepsilon\}$$
length of shortest curve.

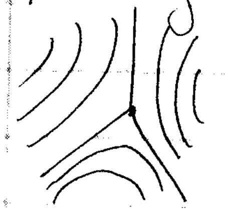
Mumford showed that these sets $M_\varepsilon(S)$ are compact.
S closed surface of genus ≥ 2

Measured foliation of a surface S
Decompos of S into 1 dimension submanifolds
+ a finite set of singular pts.

non-singular point

transition maps are
y is const $\quad (x,y) \mapsto (f(x,y), y+c)$

At sing point



p-prong singularity

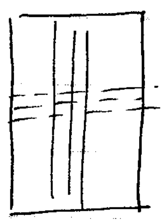
$p=3$

A measured foliation (F, μ)

(singular) foliation → F; μ ← transverse measure

Ex: Foliation of T^2 are either rational or irrational.

<u>xy structure</u>



two obvious foliations

$$(F_x, |dy|) \text{ and } (F_y, |dx|)$$

xy structure - Identify finite # of rectangles via isom of the bdry

~ surface w/2 transverse measured foliations

Gauss - Bonnet : $\sum D(p) = -2\pi \chi(S)$

Ex: Angle defect is π at each vertex

$D(p) = A(p) - 2\pi$

<u>Neilson-Thurston classification of mapping classes</u>

- Periodic (of finite order)
- reducible

  fixes a collection of distinct disjoint isotopy classes of simple closed curves
- f is pseudo - Anosov, i.e.

  $\exists$ a representative homeomorphism φ and a pair of transverse measured foliations

$$\varphi(F^u, \mu_u) = (F^u, \lambda \mu_u)$$

$$\varphi(F^s, \mu_s) = (F^s, \lambda^{-1} \mu_s)$$

Part II

(X, f) pt of Teich(S) ; equip X with structure

move in Teich space by stretching in horiz. dir. & contracting in vertical direction.

Def: Such a map is a Teichmüller map.

Teich's existence theorem: S closed, $((X,f), (Y,g))$ then $\exists$ a Teich map between (X,f) & (Y,g).

Teich's uniqueness theorem: If $g: X \to Y$ is a Teich map & $g' \sim g$ then $K(g') \geq K(g)$ (or $\exists$ conf map) with equality iff $g' = g$. iq

Corollary: Teich maps generate geodesics in Teich space.

Proof:

Let $g \in \mathrm{Isom}(\mathrm{Teich}(S))$ $\tau(g) = \inf\limits_{X \in \mathrm{Teich}(S)} d(X, gX)$

3 types of isometries:

1. $\tau(g)$ is not achieved : parabolic.
2. $\tau(g)$ is achieved $=0$ elliptic.
3. $\tau(g)$ is achieved >0 hyperbolic

Facts about teich space: • $\exists$ constant $\delta = \delta(S)$ s.t. for $X \in$ Teich any 2 closed geodesic of length $< \delta$ on X are disjoint

- For compact $A \in \mathrm{Teich}(S)$, $\exists K$ s.t. $\forall X, Y \in A$

$$\frac{1}{K} \leq \frac{l_X(C)}{l_Y(C)} \leq K \quad , \forall \text{ curve } C \in S$$

Case 1: assume $\tau(f)$ is not achieved. Let $\{X_n\} \subset \mathrm{Teich}(S)$ s.t. $d(X_n, f X_n) \Longrightarrow \tau(f)$

Suppose when we project to $M(S)$, stays in comp. set

$\exists g_n$ s.t. $g_n X_n$ stays in a comp. set of Teich(S).
$\Rightarrow$ $\exists$ convergent subseq $\cancel{Y_n} \rightarrow Y$.

$$d(Y, g_n f g_n^{-1} Y) \leq \overbrace{d(Y, Y_n)}^{\rightarrow 0} + d(Y_n, g_n f g_n^{-1} Y_n) +$$
$$+ \underbrace{d(g_n f g_n^{-1} Y_n, g_n f g_n^{-1} Y)}_{0} \text{ (b/c isometries)}$$

$$\underbrace{d(Y_n, g_n f g_n^{-1} Y_n)}_{\tau(f) \text{ we know}} = d(g_n^{-1} Y_n, f g_n^{-1} Y_n) = d(X_n, f X_n) \rightarrow \tau(f)$$

$\cancel{\tau}$

so projections can't stay in a compact set.
$\Rightarrow$ exists every compact set of Moduli space.
$\Rightarrow$ (by Mumford's compactness theorem) the length of some curve $\rightarrow 0$.

Let A be ball of radius $\tau(f)+1$. By prev. theor.
$\exists$ K s.t. $\ell_X(c) \leq K \ell_Y(c)$ in this ball $(X, Y \in A)$
Take n big enough s.t.
$$d(X_n, f(X_n)) < \tau(f) + 1 \text{ \& s.t. } \ell(X_n) < \left(\frac{1}{K}\right)^{3g-3} \cdot \delta$$

Let C_0 a shortest curve in X_n. $\ell_{X_n}(f^{-1} C_0) = \ell_{f X_n}(C_0) \leq$
$\leq K' \ell_{X_n}(C_0) < \delta$

$C_0, f^{-1}(C_0), \ldots, f^{-(3g-3)} C_0$ can't be all distinct
This is a reducing set of curves.
So $\tau(f)$ not achieved $\Rightarrow f$ is reducible.

(Case 2) $\tau(f) = 0 \Rightarrow \exists X \in \text{Teich}$ s.t. $fX = X$

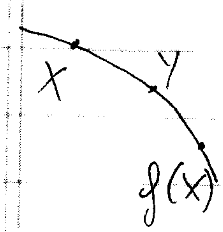
$\Rightarrow$ isometry of X $\Rightarrow$ periodic

$|\text{Isom}(S_g)| \le O\, 84(g-1)$

(Case 3) $\tau(f)$ achieved and positive.

$d(X, f(X)) = \tau(f)$

Claim f fixes the unique geodesic γ connecting X to fX.



let Y be the midpoint on this geodesic

$$\Rightarrow d(Y, fY) \le d(Y, fX) + d(fX, fY)$$

$$\qquad \frac{1}{2}\tau(f) + \frac{1}{2}\tau(f) = \tau(f)$$

$\Rightarrow f(Y)$ has to be on the geodesic. etc.,

by the same argument $f^i(X)$ lies on same geodesic

$\Rightarrow f(\gamma) = \gamma$.

So you move along a geodesic by Teich. maps.

f takes the measured ~~lamination~~ foliation defining the geodesic γ to the one with the measures changed

$\Rightarrow$ p-Anosov

[See Notes of Farb & Margulit]

Theorem (Thurston)

$$M_\varphi = S \times [0,1] \Big/ (X,0) \sim (\varphi(x), 1)$$

If • $[\varphi]$ is periodic $\Rightarrow M_\varphi$ admits a geometric structure of $\mathbb{H}^2 \times \mathbb{R}$

• $[\varphi]$ is reducible $\Rightarrow M_\varphi$ contains an (immersed) incompres. torus.

• $[\varphi]$ is p-Anosov $\Rightarrow M_\varphi$ is hyperbolic.

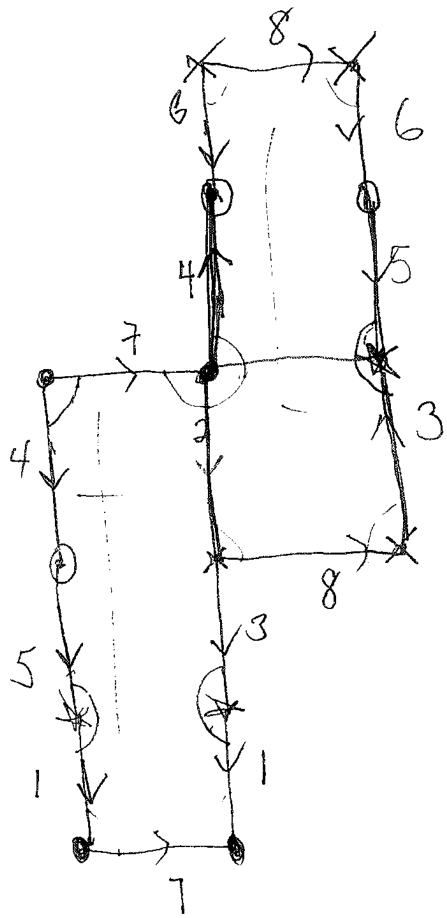
Projective classes of measured foliations give a natural boundary of Teichmüller space.

A parabolic will fix 1 pt. on the bdry, an elliptic will fix some point in the interior.

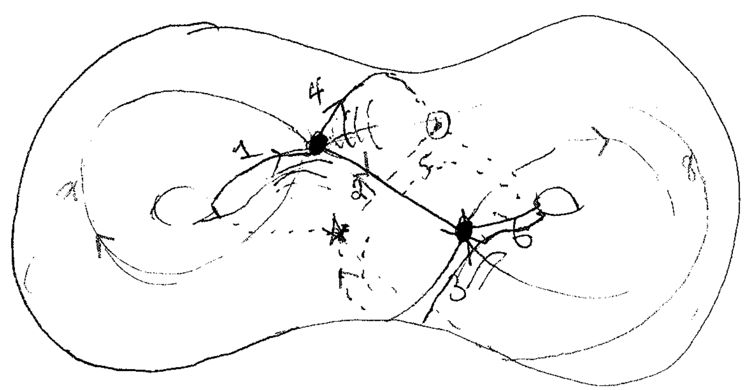
(~ conformal structure)

✱ This needs to go on the board.

put picture for board.



vertical foliation.



Ex: think about the horizontal foliations.

Gauss-Bonnet

$$\sum D(p) = -2\pi\chi(S)$$

regular

$$D(p) = A(p) - 2\pi$$

✱ π

• π

○ π

✱ X π

$\chi(S) = -2$

genus 2.