

Almost Normal Subgroups of Arithmetic Groups and the structure of Totally Disconnected Groups.

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Based on joint work with George Willis

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Theorem 1. (Margulis Normal Subgroup 1975) *Let Γ be an “arithmetic” group of “rank” ≥ 2 . Then every normal subgroup $\Lambda \trianglelefteq \Gamma$ is either finite or has finite index.*

Example 2. 1. $\gamma = SL_n(\mathbb{Z})$ (rank = $n - 1 \geq 2$)

2. $\gamma = SL_n(\mathbb{Z}[\frac{1}{p}])$ ($\mathbb{Z}[\frac{1}{p}] = \{\frac{m}{p^k} | m \in \mathbb{Z}\}$)

3. $\gamma = SL_2(\mathbb{Z}[\sqrt{d}])$ ($\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d}\}$ for $d > 0$)

We shall work with these examples throughout the talk. Later we will say what they represent and what they do not.

Definition 3. 1. Two subgroups $\Lambda_1, \Lambda_2 < \Gamma$ are called commensurable if $[\Lambda_i : \Lambda_1 \cap \Lambda_2] < \infty$ for all i .

2. $\Lambda < \Gamma$ is commensurated if $\forall \gamma \in \Gamma \ \gamma\Lambda\gamma \sim_C \Lambda$. Denote this by $\Lambda <_C \Gamma$.

3. $\Lambda < \Gamma$ is almost normal if $\exists \Lambda_0 \trianglelefteq \Gamma$ such that $\Lambda \sim_C \Lambda_0$

A trivial consequence of the definitions is $\Lambda < \Gamma$ is almost normal implies $\Lambda <_C \Gamma$.

Example 4. $SL_n(\mathbb{Z}) <_C SL_n(\mathbb{Q})$ (and $SL_n(\mathbb{Z}) <_C SL_n(\mathbb{Z}_p)$) but the subgroup is not almost normal.

Question 5 (Zimmer’s Question). (’79) *Is every $\Lambda <_C SL_n(\mathbb{Z})$ finite or finite index? We can naturally extend this question to cover other arithmetic groups.*

- Previously the unique work on the problem was by Venkataramana in 1984 which states that there is a positive answer for some groups by “brute force”. An algebraic proof of Margulis’ Theorem pushing it further. This covers $SL_{n \geq 3}(\mathbb{Z})$, but does not cover $SL_2(\mathbb{Z}[\sqrt{d}])$.
- Margulis’ proof is not algebraic. It is based upon realizing arithmetic groups as lattices (discrete subgroups of finite co-volume) in algebraic groups (and then ...).
- Our approach differs greatly from Venkataramana’s, but still does not make use of the theory of lattices. This approach exhibits new phenomenon and reduces to Margulis.

Locally Compact Totally Disconnected Groups

A locally compact group is totally disconnected if it is totally disconnected as a topological space (every connected component is a point).

Example 6.

Any discrete group (\mathbb{Z}, \dots)

Automorphism group of a graph (tree \dots) and its closed subgroups in the compact-open topology.

$SL_n(\mathbb{Q}_p)$ where \mathbb{Q}_p is the field of p -adic numbers.

A basic structural fact is that any such G has a compact open subgroup $K < G$, (e.g. stabilizer of vertex in graph, $SL_n(\mathbb{Z}_p)$) K is a profinite group (“Captures” all the topology of G). K is finite if and only if G is a discrete group.

The following is the most general way to construct profinite groups.

- Take a discrete group Γ and a decreasing sequence of finite index normal subgroups $\Gamma_i \trianglelefteq \Gamma$.
- Define the topology (metric) on Γ where γ is “close” to the identity if and only if $\gamma \in \Gamma_i$ for large i .
- Let $\hat{\Gamma}$ be the completion of Γ . Then $\hat{\Gamma}$ is a profinite group.

Margulis’ strategy for Zimmer’s question (‘80) was as follows. Given $\Lambda <_C \Gamma$ repeat a similar construction of $\hat{\Gamma}$ where topology is determined by the finite intersections of $\{\gamma\Lambda\gamma^{-1}\}$.

Exercise 7. *Let $G = \hat{\Gamma}$ be a topological (compact if and only if $[\Gamma : \Lambda] < \infty$) locally compact group. $K = \bar{\Lambda} < G$ is compact open (assuming G Hausdorff, otherwise \dots).*

Margulis’ strategy was to study dense embeddings of Γ ($SL_n(\mathbb{Z})$) into locally compact totally disconnected groups. This yields a positive answer if and only if every embedding is discrete.

Theorem (Main Theorem). *(Shalom-Willis) For all homomorphism $\varphi : SL_n(\mathbb{Z}) \rightarrow G$ where G is locally compact totally disconnected there exists a compact open subgroup $K < G$ normalized by $\text{Image}(\varphi)$.*

Thus there are no “interesting” homomorphisms from Γ to locally compact totally disconnected groups. The reason why is that we can always put $\Gamma \rightarrow \Gamma$ or $SL_n(\mathbb{Z}) \xrightarrow{\text{homo}} SL_n(\mathbb{Z}) \times \mathbb{Z}_p^n = G$.

By Margulis’ strategy, the Main Theorem is equivalent to the following theorem.

Theorem 8. *Let H be any discrete group, $\Lambda < H$ and $\varphi : \Gamma \rightarrow H$ with $\varphi(\Gamma)$ commensurating Λ . Then it normalizes some $\Lambda_0 \sim_C \Lambda$.*

Taking $H = \Gamma$ the Main Theorem reduces Zimmer's question to Margulis Normal Subgroup Theorem. This shows that the property established by the Main Theorem is strictly stronger than Zimmer'.

Example 9. $\Gamma = SL_3(\mathbb{F}_p[x])$, $H = SL_3(\mathbb{F}_p[x, x^{-1}])$, and $\Lambda = SL_3(\mathbb{F}_p[x^{-1}])$. Γ, Λ commensurate each other inside H , but do not almost normalize. Venkataramana gives a positive answer for Γ .

Another way to view the property in the Main Theorem is as a fixed point property for (conjugation) action on the space of compact-open subgroups of G . For G connected replace open by maximal: Riemannian (symmetric) space. Fixed point properties in this context is a theme of its own: Margulis Superrigidity, Harmonic maps, Gromov's random groups,

The following are the main ingredients in the proof of the Main Theorem.

1. Willis' "structure theory" of locally compact totally disconnected groups.
2. Bounded Generation of $SL_{n \geq 3}(\mathbb{Z})$ relying on analytic number theory. This is particularly involved for $SL_2(\mathbb{Z}[\sqrt{d}])$. This is due to Carter-Keller (and Paige)
3. Special algebraic features of Γ . The major challenge is the following: let $\Gamma < SL_3(\mathbb{R})$ be a discrete co-compact subgroup (Margulis implies this is necessarily an arithmetic group). Does Γ enjoy the same Main Theorem? There is no algebraic proof known for Margulis Normal Subgroup theorem for such a Γ .

Willis Theory of locally compact totally disconnected groups

Definition 10. Let G be a locally compact totally disconnected group and $B(G)$ the space of compact open subgroups (G -space by conjugates). Fix $\alpha \in \text{Aut}(G)$ (e.g. $\alpha_h(g) = hgh^{-1}$).

1. The scale of α , $S(\alpha)$ is defined by

$$S(\alpha) = \min\{[\alpha(k) : k \cap \alpha(K)] \mid K \in B(G)\}$$

2. A subgroup $K \in B(G)$ is tidy for α if

$$S(\alpha) = [\alpha(K) : K \cap \alpha(K)]$$

$S(\alpha)$ is analogous to the product of $|\text{eigenvalues}| > 1$ in matrix groups.

Definition 11. $L < \text{Aut}(G)$ is called flat if $\exists K \in B(G)$ which is tidy $\forall \alpha \in L$. Note that $L(1) = \{\alpha \in L \mid S(\alpha) = 1 = S(\alpha^{-1})\}$ normalizes K .

Willis: $L(1) \trianglelefteq L$ and $L/L(1) \cong \mathbb{Z}^r$ (which implies $L(1) \geq [L, L]$).

Theorem 12. (Willis) Any finitely generated abelian $L < \text{Aut}(G)$ is flat.

Theorem 13. (Shalom-Willis) Similarly for nilpotent and polycyclic (but not all solvable) groups.

Proof. (Sketch of Main Theorem)

1. Take

$$L = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

where Γ is nilpotent. L is flat by the above theorem.

$$[L, L] = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

normalizes some $K_0 < G$. Similarly for any elementary subgroup $L^1 < \Gamma$.

2. By bounded generation Γ is a product of 1000 such subgroups hence:

$$(*) [\alpha(K_0) : \alpha(K_0) \cap K_0]$$

bounded over $\gamma \in \Gamma$.

3. (*) implies Γ normalizes some $K'_0 \sim_C K_0$.

□

Return to $SL_n(\mathbb{Z}[\frac{1}{p}])$ for $n \geq 2$. It does commensurate $SL_n(\mathbb{Z})$.

Theorem 14. If S is any set of primes $\mathbb{Z}[S]$ =localization by primes in S , and $\Lambda <_C SL_n(\mathbb{Z}[S])$ then: $\exists S' \subset S$ such that $\Lambda \sim_C SL_n(\mathbb{Z}[S'])$.

A similar theorem holds for \mathcal{O} =ring of integers with ∞ units (e.g. $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$ $d > 0$). A problem that is probably out of reach would be to try to prove the same theorem for $\Gamma = Sp(n, 1)(\mathbb{Z})$. This group is not boundedly generated, and does not satisfy Margulis Normal Subgroup Theorem and thus lies between rigid and non-rigid arithmetic groups.