When mr. Artin leaves

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Before leaving

W: finite, irreducible Coxeter group.

$\{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{R}^d$: positive roots of the associated Coxeter system.

With $H_i := \{z \in \mathbb{C} \mid \langle z \mid \alpha_i \rangle = 0\}$, we call $\mathcal{A}_W := \{H_1, \ldots, H_n\}$ the

Coxeter Arrangement of type W .

Let

$$Y_W := \mathbb{C}^d \setminus \bigcup_{i=1}^{n} H_i, \quad X_W := Y_W/W.$$

... de côté pour considérer la cohomologie des groupes G_W et H_W. On espère pouvoir calculer la cohomologie de ces groupes à cause de la conjecture suivante :

CONJECTURE.- Les espace X_W et Y_W sont des espaces d'Eilenberg-MacLane.

Malheureusement je n'ai pu prouver cette conjecture que pour quelques types de groupes de réflexions W. (Dans ce qui suit la notation des différents ...

THE GROUP $\pi_1(\mathcal{A})$

Let $\pi_1(\mathcal{A}) := \pi_1(\mathcal{M}(\mathcal{A}))$.

Theorem. [Arvola '92] *There is a finite presentation of* $\pi_1(\mathcal{A})$.

In general however, Arvola's presentation is not very handy.

Key-names: Cohen, Denham, Papadima, Randell, Salvetti, Suciu, Schenk.

Theorem. [Randell '02] *The space* $\mathcal{M}(\mathcal{A})$ *has the homotopy type of a (fnite) minimal CW-complex.*

If $\mathcal{A}$ is *complexified*, this minimal complex can be explicitely constructed... in two ways! [Salvetti & Settepanella '07; D. '07]

When is $\mathcal{M}(\mathcal{A})$ a $K(\pi_1(\mathcal{A}), 1)$?

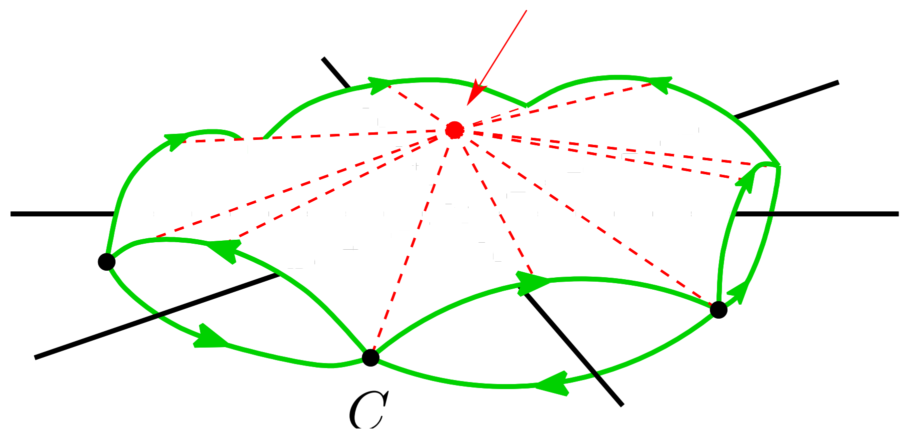
The Salvetti complex $Sal(\mathcal{A})$

[Salvetti '87]

Given a complexified arrangement $\mathcal{A}$, consider the oriented graph $\Gamma(\mathcal{A})$ with:

$V(\Gamma(\mathcal{A}))$: chambers of $\mathcal{A}$

$E(\Gamma(\mathcal{A}))$ contains (C_1, C_2) iff C_1 is adjacent to C_2.



- $Sal(\mathcal{A})^1 = \Gamma(\mathcal{A})$
- A k-cell $[F, C]$ for every codimension-k face F and every chamber $C < F$.
- $[F, C]^1$: all oriented *(positive)* paths starting at C and passing exactly once *(minimal)* through all $H \supset F$.

Theorem. [Salvetti '87] *If $\mathcal{A}$ is complexified, $Sal(\mathcal{A}) \simeq \mathcal{M}(\mathcal{A})$.*

Coverings of $Sal(\mathcal{A})$

$Sal(\mathcal{A})$ is a regular CW-complex. Thus:

- One obtains presentations of the fundamental group.
- If $\mathcal{S}$ is the poset of cells of $Sal(\mathcal{A})$, then $\Delta(\mathcal{S}) \simeq \mathcal{M}(\mathcal{A})$.

Theorem. [D. '03] *For every complexified arrangement $\mathcal{A}$ and every covering $\rho : \mathcal{M}_\rho \to \mathcal{M}(\mathcal{A})$ there is a poset $\mathcal{S}_\rho$ with*

$$\Delta(\mathcal{S}_\rho) \simeq \mathcal{M}_\rho.$$

Remark.

- *$\mathcal{S}_\rho$ is the poset of cells of a regular CW-complex.*
- *If $\mathcal{A}$ is a reflection arrangement and ρ the universal covering, this specializes to the construction of Davis and Charney.*

The arrangement groupoid

(Again:) $Sal(\mathcal{A})$ is a regular CW-complex.

Definition. *The arrangement groupoid $\mathcal{G}$ is the instance of the Poincaré groupoid of $\mathcal{M}(\mathcal{A})$ corresponding to $Sal(\mathcal{A})$.*

(Objects of $\mathcal{G}$ are vertices of $Sal(\mathcal{A})$, and morphisms are homotopy classes of paths on Γ).

Remark. *Inside $\mathcal{G}$ sits the* category of positive paths $\mathcal{G}^+$.

Recall the covering theory of groupoids.

Theorem. [Folkore] *To every topological covering $\mathcal{M}_\rho$ of $\mathcal{M}(\mathcal{A})$ corresponds a covering of groupoids $\rho : \mathcal{G}_\rho \to \mathcal{G}$ with the same characteristic group.*

THE OTHER WAY AROUND

Definition. *Given a covering $\rho : \mathcal{G}_\rho \to \mathcal{G}$, define the flag complex U_ρ:*

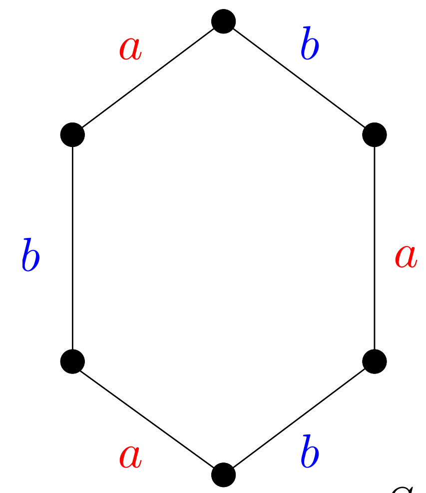
- *vertex set :* $\mathrm{Ob}(\mathcal{G}_\rho)$,
- *d-simplices : $(d+1)$-sets $\{\gamma_0, \ldots, \gamma_d\}$ such that there are positive minimal paths $\alpha_1, \ldots, \alpha_d$ with:*
  (i) $\gamma_{i+1} = \gamma_i \alpha_i$ for all $i = 1, \ldots, d$, and
  (ii) $\alpha_1 \ldots \alpha_d$ positive minimal.

- If ρ is the universal cover of a real reflection arrangement, then U_ρ is a known 'Garside-type' complex. [Bestvina '99]

Proposition. [D. '05] *For any linear complexified arrangement $\mathcal{A}$ and any topological covering $\rho : \mathcal{M}_\rho \to \mathcal{M}(\mathcal{A})$ there is a homotopy equivalence*

$$U_\rho \simeq \mathcal{M}_\rho.$$

Combinatorics of Garside structures



Data:

- A partially ordered set P

(a bounded, graded lattice)

- A labeling λ of the edges of the Hasse diagram of P in some alphabet S .

(balanced, group-like)

Definition. *The Garside group* $G(P, \lambda)$ *is generated over* S *and defined by the relations given by identifying every two words corresponding to unrefinable chains with same begin- and endpoint.*

Example. In our situation, $G(P, \lambda) = \langle a, b \mid aba = bab \rangle$.

Definition. *The complex* $X(P, \lambda)$ *is the flag complex with vertex set* $G(P, \lambda)$ *and an edge connecting* g *with* h *if* $g^{-1}h$ *is one of the words that can be read off* P *along an unrefinable chain.*

Order on chambers

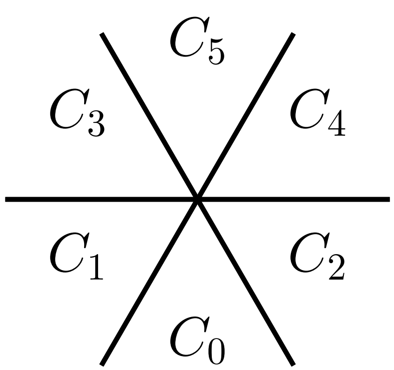
Definition. *Let C_1, C_2 be chambers of $\mathcal{A}$.*
$S(C_1, C_2) \subset \mathcal{A}$: *the set of hyperplanes separating C_1 from C_2.*
Fix a chamber C_0. The partial order

$$C_1 \leq_{C_0} C_2 \Leftrightarrow S(C_0, C_1) \subseteq S(C_0, C_2)$$

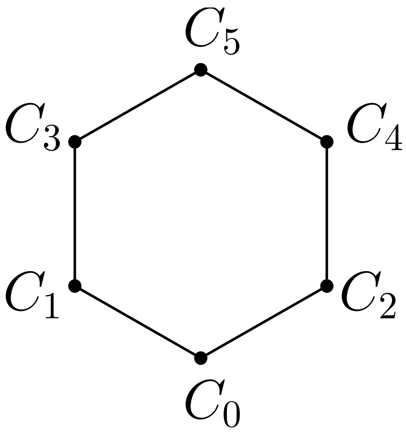
defines the poset of regions $\mathcal{P}_{C_0}(\mathcal{A})$ based at C_0.

For $\mathcal{A}$ linear, the Hasse diagram of this poset 'is' a plane projection of the 1-skeleton of the polar polytope of the fan given by $\mathcal{A}$.

$\mathcal{A}$:



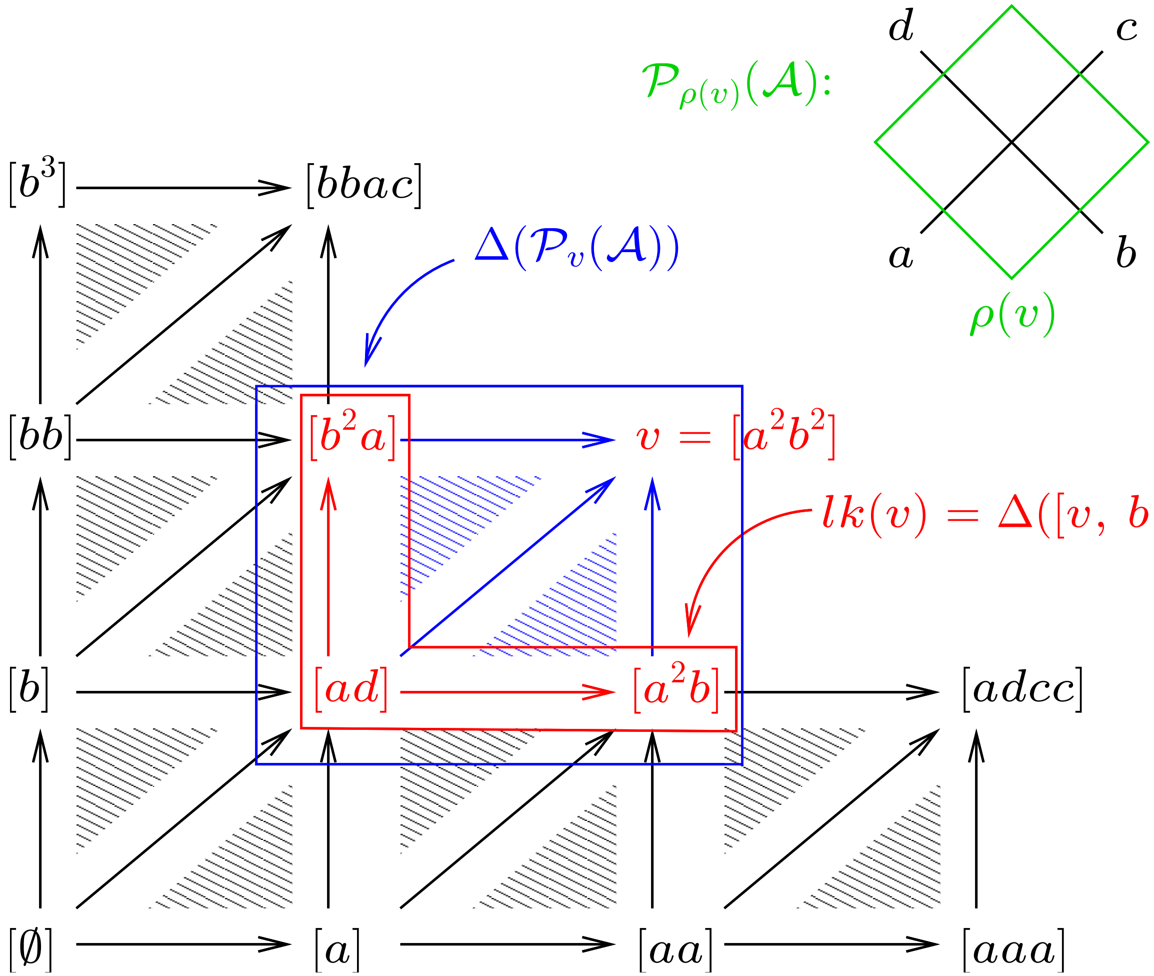
$\mathcal{P}_{C_0}(\mathcal{A})$:



THE POSITIVE COMPLEX

Let $\hat{U}^+$ be the subcomplex spanned by the positive paths.

$\mathcal{P}_{\rho(v)}(\mathcal{A})$:



Note:

If $lk(v) = \Delta([v, w]_{>v})$ for $w \in \mathcal{P}_v(\mathcal{A})$, then $\hat{U}_n^+$ retracts onto $\hat{U}_{n-1}^+$.

DELIGNE'S THEOREM

The condition $lk(v) = \Delta([v, w]_{>v})$ for $w \in \mathcal{P}_v(\mathcal{A})$ means that

> for every positive path v there is a unique positive path w_v so that if the class v can be written $v = [\beta\alpha]$ for β, α positive and α minimal, then β is a representative of a class $[\beta] \leq_{\rho(v)} w_v$, i.e., that 'ends' in the interval $[v, w_v]$.

this is the so-called property D .

Here is the *"chef d'oeuvre"*:

Theorem. [Deligne '72] *If every chamber of $\mathcal{A}$ is a simplicial cone, then property D holds, and the universal cover of $\mathcal{M}(\mathcal{A})$ is contractible.*

... and guess what?

Deligne's theorem

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Theorem. [Björner, Edelman and Ziegler '92; Paris '96; D. '06] *The chambers of $\mathcal{A}$ are simplicial cones if, and only if $\mathcal{P}_C(\mathcal{A})$ is a lattice for every base chamber C.*

Inventiones math. 17, 273–302 (1972)

Les immeubles des groupes de tresses généralisés

Pierre Deligne (Bures-sur-Yvette)

Introduction

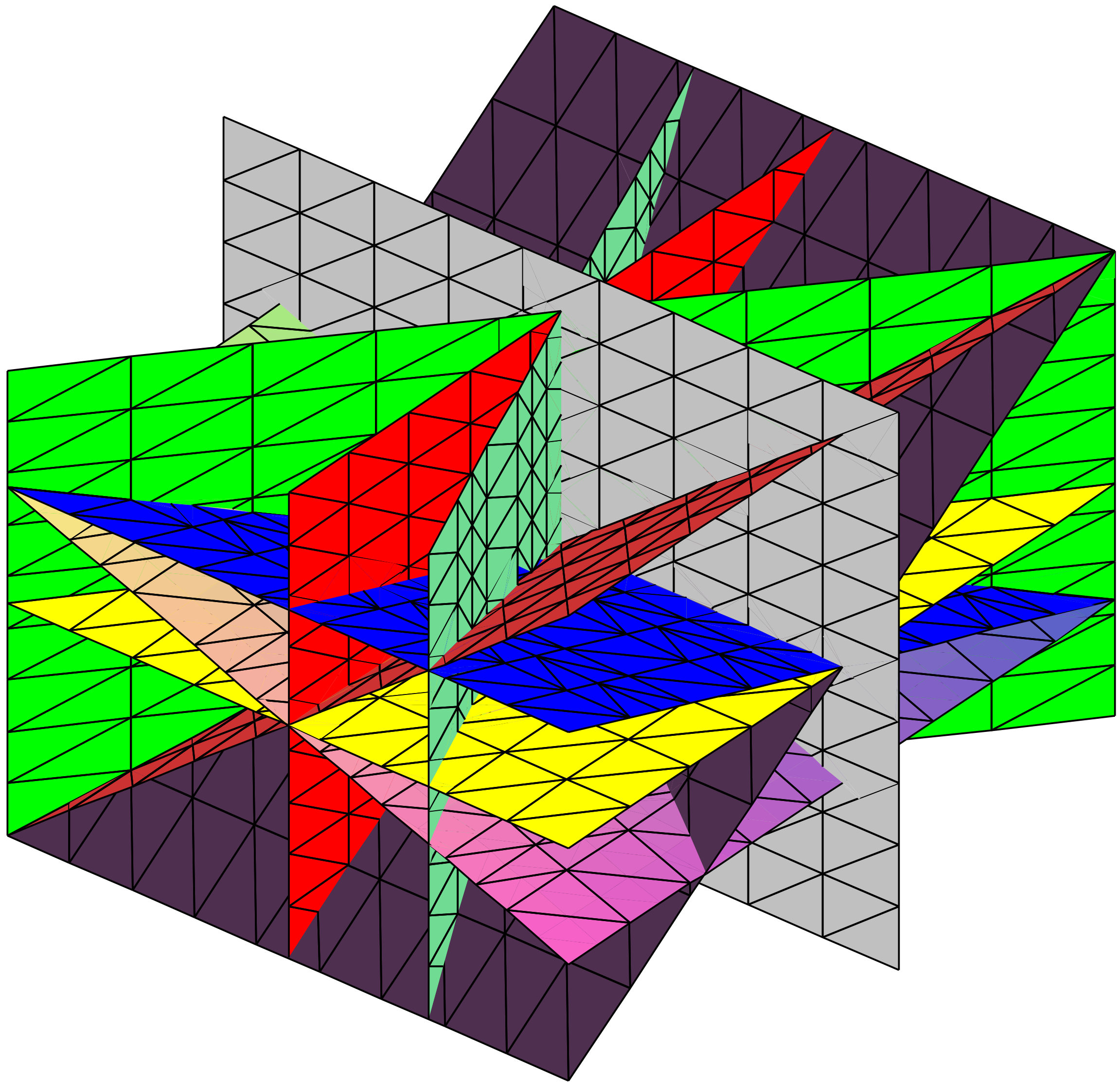
Le résultat principal de ce travail est le suivant.

Théorème. *Soit V un espace vectoriel réel de dimension finie, $\mathcal{M}$ un ensemble fini d'hyperplans homogènes de V, $V_{\mathbb{C}}$ le complexifié de V et $Y = V_{\mathbb{C}} - \bigcup_{M \in \mathcal{M}} M_{\mathbb{C}}$. On suppose que les composantes connexes de $V - \bigcup_{M \in \mathcal{M}} M$ sont des cônes simpliciaux ouverts. Alors, Y est un $K(\pi, 1)$.*

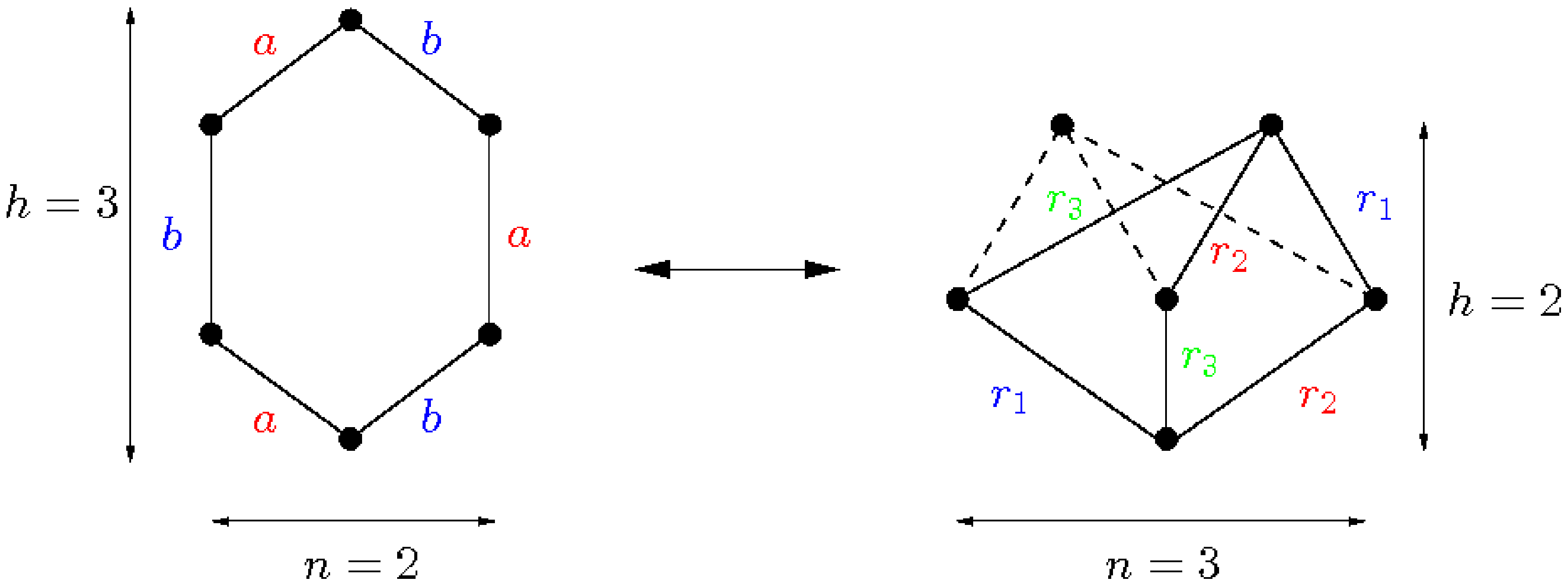
Soient V comme plus haut, et $W \subset GL(V)$ un groupe fini engendré par des réflexions. On suppose qu'aucun vecteur non
fixe sous W:

La description de $\mathcal{S}$ et la possibilité de c) étaient apparues lors d'une conversation avec Brieskorn et Tits au printemps 1970. Les idées requises pour établir b) m'ont été fournies par Garside [4], que je suis souvent de très près.

hypothèse
sur l'espace Y_W correspondant.



Dual Garside structures



$$\langle a,b \mid aba = bab\rangle \simeq \langle r_1, r_2, r_3 \mid r_1r_3 = r_3r_2 = r_2r_1\rangle$$

"Weak order ↔ (generalized) Noncrossing Partitions"

Finite complex reflection groups

The nice thing about dual Garside structures is that they generalize in the complex setting .

For a finite real reflection group W let (P',λ') and (P'', λ'') be the two associated Garside structures.

Theorem. [Bessis '06] *It can be proved directly that $X(P'', \lambda'')$ is the universal covering of the orbit space of $\mathcal{A}_W$.*

Moreover, if W is a finite complex reflection group, "something like $X(P'', \lambda'')$" is the universal cover of the orbit space of $\mathcal{A}_W$.

In particular, every finite complex reflection arrangement is $K(\pi, 1)$.

(and in particular, those complexes are finite $K(\pi, 1)$s for the finite complex reflection groups).

But: Still little understanding of the geometry!