Residual Finiteness & Sep. of q-convex subgps.
(r.f.)

Manning
jt with Agol, Groves.

Def ① G is r.f. if $\forall g \in G \; \exists \; G \xrightarrow{\varphi} F$ finite so $\varphi(g) \neq 1$.

Ex ① (Mal'cev) Linear.
② $G = \mathrm{Aut}(H)$, H r.f.

$\Rightarrow$ (if G fp) sol'n to word problem.

② $H < G$ is separable if $\forall g \in G \setminus H \; \exists \; G \xrightarrow{\varphi} F$ finite so $\varphi(g) \notin \varphi(H)$.

$\Rightarrow$ can solve membership problem

Ex ① (Hall) Free gps are LERF (= all subgps separable.) (Locally extended residually finite)

② (Scott) Surface gps are LERF

③ (Agol-Long-Reid) Bianchi gps are GFERF.

④ (Wise, Haglund-Wise, ...) Lots of 3-mfld gps.

Q0: Does every hyp gp have a finite quotient? (hyp = word hyp)

Q1: Is every hyp gp RF?

Q2: Is every hyp gp QCERF? (q convex subgps are sep.)

(Kapovich-Wise) Q0 and Q1 are equivalent

JM2

<u>Thm 1 [AGM]</u> If every hyp gp is RF, then every hyp gp is QCERF

<u>Remarks</u>: ① "Guessed" by Wise
② You can take this either way. (either showing all hyp. are RF gives you much more
or, finding qconvex subgp not sep. & gives place to look for not RF)

<u>Def</u> Given $H < G$, we say $\{g_1, \ldots, g_k\}$ is <u>essentially distinct</u> (WRT H) if $g_i H \neq g_j H$ when $i \neq j$.

In this case, say $H^{g_1}, \ldots, H^{g_k}$ are <u>essentially distinct</u>
(doesn't actually mean they're different!)

The <u>height</u> of H is the max # of essentially distinct $H^{g_1}, \ldots, H^{g_k}$ st $\bigcap_i H^{g_i}$ is infinite.

<u>Ex</u>: $H \trianglelefteq G \Rightarrow$ height = index.

<u>Thm</u> (GMRS) Height(H) is finite if $H^{QC} < G^{hyp}$

<u>Thm 2</u> (AGM) Suppose G is (torsion free), hyp, RF. Let $H < G$ be $\overset{QC}{q\text{-convex}}$.
Let $g \in G \setminus H$
There is a quotient $G \xrightarrow{\eta} \bar{G}$ st ① $\bar{G}$ is hyp, ② $\eta(g) \notin \eta(H)$
③ $\eta(H)$ is QC
④ height of $(\eta(H)) <$ height(H).

Pf that Thm 2 $\Rightarrow$ Thm 1.

$H^{QC} < G^{\text{hyp, rf} \leftarrow \text{by assumption}}$, want to show H (is profinitely closed) is separable.

Lemma 1. $\exists\, G_0 \subset G$, finite index and tf.

Lemma 2 $H_0 = H \cap G_0$ is sep in $G_0 \Leftrightarrow H$ sep in G.

Induct on height(H).

Base Case H finite ✓ (RF for G)

Inductive step: WLOG, assume G tf. Choose $g \in G \setminus H$.

$$\begin{array}{ccc} g \notin H & \hookrightarrow & G \\ \downarrow & & \downarrow \eta \\ \eta(g) \notin \eta(H) & \hookrightarrow & \overline{G} \\ \downarrow & & \downarrow \varphi \\ \varphi(\eta(g)) \notin \varphi(\eta(H)) & \hookrightarrow & F \text{ (finite.)} \end{array}$$

■

Pf of Thm 1

Use finite height (=k) of H to build peripheral structure for H and then G.

- Let $\mathcal{D}_0 = \{$minimal, ∞ intersections $\bigcap_{i=1}^{k} H^{g_i}$ with $g_1 = 1$, $\{1, g_2, \dots, g_k\}$ ess. distinct$\}$

  (maybe smaller intersections with fewer g_i than this, but we don't care about those.)

- $\mathcal{D}_0 \rightsquigarrow \mathcal{D}$ · Replace $D \in \mathcal{D}_0$ by $\mathrm{comm}_H(D)$
  · Choose one per H-conj. class.

- $\mathcal{D} \rightsquigarrow \mathcal{P}$ · Replace $D \in \mathcal{D}$ by $\mathrm{comm}_G(D)$
  · Choose one per conj. class (in G)

Ex $G = \langle a, b \rangle$, $H = \langle a^2, ba^2b^{-1} \rangle$

$$\mathcal{D} = \{ \underset{H \cap H^{b^{-1}}}{\langle a^2 \rangle}, \underset{H \cap H^{b}}{\langle ba^2b^{-1} \rangle} \}$$

$$\mathcal{P} = \{ \langle a \rangle \}.$$

Prop ① H is hyp & rel to $\mathcal{D}$ — $(H, \mathcal{D})$ is rel hyp

② $(G, \mathcal{P})$ is rel. hyp.

③ $(H, \mathcal{D})$ is relatively q-convex in $(G, \mathcal{P})$.

Strategy: Apply Dehn filling

New bits: Given a λ-rel q-convex subgp and long enough "H-filling" of $(G, \mathcal{P})$, can ensure that:

① Image of H rel. Q-convex.

② $\eta(g) \notin$ Image of H

③ If filling kernels are finite index, then height goes down.