Groups with f.p. properties

Januszkiewicz

G. Arzhantseva, M Bridson - T J - I. Leary, A Minasyan, J. Świątkowski.

TJ - P. Kropholler - I.L.

Serre's exercise.

(W, S) a Coxeter gp. When does it have a fp free action on a tree?

Relations: $s^2 = 1$, $(st)^{m_{st}} = 1$: If one m_{st} is ∞, then W splits

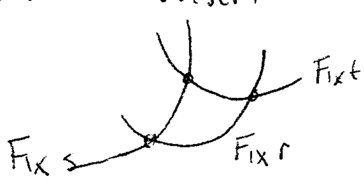
$$W = W_{S-\{t\}} \underset{W_{S-\{s,t\}}}{*} W_{S-\{s\}} .$$

What happens when all m_{st} finite?

Then any action on T has a fixedpoint.

- $W_{\{s\}}$ has nonempty $\mathrm{Fix}\{s\}$ ($v \mapsto s.v \mapsto v$ so midpt of geodesic is fixed)
- $\mathrm{Fix}\{s\} \cap \mathrm{Fix}\{t\} = \mathrm{Fix}\{W_{st}\}$
- Helly implies that $\bigcap_{s \in S} \mathrm{Fix}(s) \neq 0$.
  - Every 3 gens share a fp.

Suppose not: $W_{\{str\}}$ has empty fp set.



Fix are convex sets; pairwise intersection.

$H^1(\bigcup \mathrm{Fix}_{st}) \neq 0$,

contradiction.

2 TJ

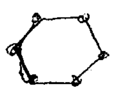
Generalize this? Action of Coxeter gps on other spaces? Need convexity - how about CAT(0)?

Farb · Barnhill · (Bridson):

(W, S) st $W(s_{i_1}, \ldots, s_{i_k})$ finite group.

then any isometric action on CAT(0) space of dim. ~~k~~ $k-1$ has a fp.

Pf similar to tree. intersection of convex sets give nontrivial homology in dim k, bad.

<u>Ex</u> $\tilde{A}_n$  $n+1$ gen $\tilde{A}_n \curvearrowright \mathbb{R}^n$.

Any isometric action on X^{n-1} CAT(0) space has a fp.

(In T.J., P.K-I.L): If Q is a common quotient of <u>all</u> $\tilde{A}_n$, then Q has a fp for any isometric action on any fin dim CAT(0) space.

Common quotients trivial. $\mathbb{Z}_2$ ~~...~~ No others?

[A M Osin] [Olshanskii]: Given $\{G_n\}$ nonelementary hyp; H, any countable gp, Then $\exists Q$ infinite, $G_n \twoheadrightarrow \underline{Q \hookleftarrow H}$ $\forall n$

Q simple, Kazhdan (Q periodic)

~~...~~

Every $G_n \to Q_n \hookleftarrow H$, Q_n are rel H hyp.

There is a seq on G_n non elem., hyp. st G_n is gen by $n+1$ elts, every n-elt subset spans a finite gp.

(Helly argument) thus G_n has a f_p on any CAT(0) space of dim $<n$.

<u>Smith Theory</u> X <u>p-acyclic</u>, fin dim, G finite p-gp.

$\tilde{H}^*(X; \mathbb{Z}/p\mathbb{Z}) = 0$.

then $X^G \neq \emptyset$. X^G is p-acyclic.

(has more to it, but this is all we need)

(*) Suppose $\{G_n\}$ is non eltary hyp gen by $(n+1)$-elts, and every k-elt set $k < n+1$ spans a finite p-gp. Then

G_n acting on X^{n-1}, p-acyclic, $\dim X < n$ has a f_p.

hence any common quotient Q of G_ns has a f_p for any action on a fin dim'l v space p-acyclic

<u>Ex</u> $Q \curvearrowright Q$ by translations ~~non~~ effective.

<u>Remark</u> If Q (from (*)) acts simplicially on a locally finite complex, fin dim, p-acyclic, contractible then the action is trivial.

<u>Pf</u> Q has a f_p. Simplicial $\Rightarrow$ stabilizes combinatorial balls $B(r, x)$, hence is an identity.

- If Q acts C^ω on a contractible f. dim mfld, then the action is trivial (Pf as above - use ~~[illegible]~~ formal power series)

TJ 4)

There are $\{G_n\}$ non el, hyp, gen by $(n+1)$ elts, every k elt set $k < n+1$ spans finite p-gp, G_n acting on X^{n-1}.

Developable simplices of gps G^S

S a set. $T \rightsquigarrow G_T$ $\quad$ G_T gen by T

$T \subset T'$ $\quad G_T \hookrightarrow G_{T'}$ $\quad$ to simplify these inclusions, gens $\longmapsto$ gens.

"inductively constructing bigger gps"

Given G_T, many choices for G_S. But there is universal "biggest" one $\tilde{G}_S$.

$$\begin{tikzcd} G_T \arrow[r, leftrightarrow] \arrow[d] & G_S \\ \tilde{G}_S \arrow[ur, dashed] & \end{tikzcd}$$

Ex Coxeter gps.

The space associated with $\{G_T\}$: $\Delta_S^{op} \times G_S/\sim$ $\quad$ codim 1 spaces indexed by elts of S.

$$(p,g) \sim (q,h) \Leftrightarrow p = q,\ g^{-1}h \in G(p).$$

Retractible simplex of gps $\quad r_T(t) = \begin{cases} t & \text{if } t \in T \\ \{e\} & \text{otherwise.} \end{cases}$

r_T extends to a homomorphism $r_T : G_S \to G_T$.

* loc Retractible $\Rightarrow$ developable $\Rightarrow \tilde{G}_S$ retractible.

~~[illegible]~~. Given G_T, $\exists$ many G_S, but $\exists$ ~~the~~ unique up to isom. terminal one, $\overline{G_S}$.

"Do it correctly $\Rightarrow$ $\overline{G_S}$ is direct prod."

$$\begin{tikzcd} G_T \arrow[r, shift left] \arrow[r] \arrow[r, shift right] \arrow[d] & G_S \arrow[dl, dashed] \\ \overline{G_S} & \end{tikzcd}$$

If G_T are finite, so is $\overline{G_S}$

If G_T are ~~[illegible]~~ p-gps, so is $\overline{G_S}$.