

Note Taker Checklist Form -MSRI

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Talk Title and Workshop assigned to:

Homomorphisms to mapping class groups /
Intro to Geometric Groups Theory

Lecturer (Full name): Daniel Groves

Date & Time of Event: 8/27 3:45-4:35 pm

Check List:

- () Introduce yourself to the lecturer prior to lecture. Tell them that you will be the note taker, and that you will need to make copies of their own notes, if any.
- () Obtain all presentation materials from lecturer (i.e. Power Point files, etc). This can be done either before the lecture is to begin or after the lecture; please make arrangements with the lecturer as to when you can do this.
- () Take down all notes from media provided (blackboard, overhead, etc.)
- () Gather all other lecture materials (i.e. Handouts, etc.)
- () Scan all materials on PDF scanner in 2nd floor lab (assistance can be provided by Computing Staff) – Scan this sheet first, then materials. In the subject heading, enter the name of the speaker and date of their talk.

Please do **NOT** use **pencil** or colored pens other than black when taking notes as the scanner has a difficult time scanning pencil and other colors.

Please fill in the following after the lecture is done:

1. List 6-12 lecture keywords: Mapping Class group, homomorphisms,
splitting, curve complex

2. Please summarize the lecture in 5 or less sentences.

The purpose of the talk was to state &
discuss a theorem about homomorphisms to $\text{Mod}(S)$
saying that there are finitely many sys H_i of $\text{Mod}(S)$ s.t.
either $W(\Gamma)$ is conjugate to one of them or \exists f.i.d. $P_0 \leq P$
which has a non-trivial graph of groups decomposition
s.t. homomorph. takes edge gr. to a sys for $\text{Mod}(S)$ fixing a curve
Once the materials on check list above are gathered, please scan ALL materials and send to the Computing Department. Return this form to Larry Patague, Head of Computing (rm 214)

TALK FOR INTRODUCTION TO GEOMETRIC GROUP
THEORY CONFERENCE - MSRI AUGUST, 2007

DANIEL GROVES

JIM CANNON: ‘Curvature has profound implications for the structure of groups and for algorithmic properties.’

BENSON: ‘Actually, his paper (in his book) is also amazing, and the motivation for the work I’m talking about today came partly from reading his paper.’

The title of the talk is:

‘Homomorphisms to the mapping class group’

The theorem which I hope to state towards the end of the talk is

Theorem A. *Let Mod_S be the mapping class group of a (topologically finite) orientable surface S and let $\mathcal{C} = \mathcal{C}(S)$ be the curve complex associated to S . Let Γ be a finitely presented group. There is a finite collection of subgroups $\mathcal{F}_{\Gamma,S}$ of Mod_S satisfying the following:*

If $\rho: \Gamma \rightarrow \text{Mod}_S$ is any homomorphism so that $\rho(\Gamma)$ is not conjugate to an element of $\mathcal{F}_{\Gamma,S}$ then there is a finite index subgroup $\Gamma_0 \leq \Gamma$ and a nontrivial graph of groups \mathcal{G} so that $\rho_0 = \rho|_{\Gamma_0}$ factors as $\rho_0 = \phi \circ \psi$ where $\psi: \Gamma \rightarrow \pi_1(\mathcal{G})$ is surjective and $\phi: \pi_1(\mathcal{G}) \rightarrow \text{Mod}_S$ is so that the images in Mod_S of the edge groups of \mathcal{G} each fix a curve on S (up to isotopy).

This is what I’ve been thinking about recently, and so what I want to talk about (I’d also like to get it on record that I’ve proved this theorem).

EMPHASISE THAT understanding sets of homomorphisms between groups is key to many topics in geometric group theory. (Acknowledge my debt to Zlil.) (Of course, before Zlil there was Morgan-Shalen and Culler, and Paulin, Bestvina, and of course Rips). Talk about other things I know: Isomorphism problem (and other algorithmic things), Hopf/co-Hopf property, MR diagrams, elementary theory... Particularly equations — ‘algebraic geometry over groups’.

but today we’re concentrating on a topological problem.

(NOTE: In the general context, post-conjugating the homomorphism is the same as changing the basepoint for the action – so it is natural to consider conjugacy classes of homomorphisms.)

Outline of talk:

• Apologise for talking about my latest theorem in an introductory workshop. However, my justifications for this are:

- (1) It's all I've been thinking about and I'm obsessed;
- (2) I'm not going to give any of the details of the proof, except in the simplest case, where I'll give a proof which is mostly unrelated to the one in the general case;
- (3) I'll try to bring out some themes from other things happening this week and last: Benson's talks; Lisa's talks from Womens; Cannon's talks (negative curvature); complex of curves was important in the talks by Jeff, Ken and Yair in Kleinian groups workshop...
- (4) I'll also bring out what I think is a different important theme in GGT, namely ...

• Understanding sets of the form $\text{Hom}(\Gamma, G)$. Give reasons for studying $\text{Hom}(G, H)$: (see above, and also bundles).

- (1) $H = \mathbb{R}$ – first cohomology;
- (2) $H = \text{SL}(2, \mathbb{C})$ – representation varieties;
- (3) But we're going to concentrate on H being discrete (fix H and let G be arbitrary):
 - (a) Subgroup structure of H ;
 - (b) If $G = H$ then we study $\text{Aut}(H)$, or whether H is Hopfian, or co-Hopfian;
 - (c) 'algebraic geometry over H ';
 - (d) The universal theory of H , leading to logic over groups...
 - (e) For us today, we're interested in bundles...

• Surface bundles. Define. Motivate. Classify by homotopy classes of maps, which are the same as conjugacy classes of homomorphisms. Get to Mod_S from $\text{Diff}^+(S)$ because the connected component of the identity in $\text{Diff}^+(S)$ is contractible. Therefore, of fundamental interest is understanding conjugacy classes of homomorphisms from an arbitrary group to Mod_S . (We might not go through this whole theory in the lecture, just mention that understanding bundles relates to understanding

In this section we should also mention characteristic classes and why cohomology may be less useful for understanding surface bundles than it is for vector bundles.

NOTE: I really want to understand conjugacy classes of homomorphisms, but at the moment all I have is results about homomorphic images.

- A brief detour to understand why negative curvature (in the target) helps understand homomorphic images. Explain the theorem in case the target is free (rather than Mod_S). This is easy (particularly since all subgroups of a free group is free), but it might be worth explaining how to put a lamination on the (triangular) presentation 2-complex of Γ and thereby get a splitting. THINK CAREFULLY ABOUT WHAT TO SAY HERE, because I'm not going to explain the proof of Theorem A at all. Say anything about hyperbolic groups? Limits, canonical representatives, etc.

*** Actually, let's prove the theorem in case of $SL(2, \mathbb{Z})$ – pass to a finite index free subgroup and then prove it for a free group!

- Where does the negative curvature in Mod_S come from? The curve complex. Talk a little about the work of Masur and Minsky... (or just say 'The negative curvature in the curve complex (Masur-Minsky) can be used to study the mapping class group. This is the approach that we take, and we rely heavily on the results of Masur and Minsky'.)

- State the main theorem and the corollaries about all pseudo-Anosov subgroups of Mod_S and about Property (T) groups. (Mention Bowditch for all pseudo-Anosov surface groups.)

SOMEWHERE: List the different kinds of elements of the mapping class group - talk about curve stabilisers (mapping class groups of surfaces of smaller 'complexity').

SOMEWHERE: Talk about graphs of groups and how the hope is to split the group Γ up into simpler pieces – we understand the edge groups by induction? And hopefully the vertex groups are 'simpler'? NOTE that much of this is speculative in that there is no JSJ theory yet, or MR diagrams, or a real shortening argument, or a real sense of how to do this...

AS IN BENSON'S PAPER (In his book), we have bijections

$$\left\{ \begin{array}{c} \text{Isomorphism} \\ \text{classes of } S\text{-bundles} \\ \text{over } B \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Homotopy} \\ \text{classes of maps} \\ B \rightarrow \text{BMod}_S \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Conjugacy classes} \\ \text{of representations} \\ \rho : \pi_1 B \rightarrow \text{Mod}_S \end{array} \right\}$$

(I'm taking into account torsion in Mod_S so I'm using BMod_S rather than the moduli space.)

NOTE: BMod_S is the classifying space for the group Mod_S . Since the connected component of the identity in $\text{Diff}^+(S)$ is contractible (Earle-Eells), and $\text{Mod}_S = \pi_0(\text{Diff}^+(S))$, there is a homotopy equivalence between the classifying spaces:

$$\text{BDiff}^+(S) \simeq \text{BMod}_S.$$

Motivation: To understand $\text{Hom}(G, H)$.

For ex, $H = \mathbb{R} \rightarrow H^1$

$H = \text{SL}(2, \mathbb{C}) \rightarrow$ representation varieties.

H discrete. If H acts on X and

$h: G \rightarrow H$ homomorph $\Rightarrow G$ acts on X .

$G = H$ $\text{Hom}(G, H) = \text{Aut } G$

Is G Hopfian? G is Hopfian if every surjective homomorphism is isom.

Is G co-Hopfian? (every injective is isom)

- More generally, focusing on surjective homomorphisms could lead to an understanding of sgr. of H

Equations over H : Variables corresp to group elements. If we have variables x_1, \dots, x_n , and want to understand all sol. to a system of eq. $\{w_1(x_1, \dots, x_n), \dots, w_k(x_1, \dots, x_n)\}$
A solution is a tuple $\underline{h} \in H^n$ s.t. $w_i(\underline{h}) = 1$.

Observe: $\text{Hom}(\langle x_1, \dots, x_n \mid w_1(x) \dots w_k(x) \rangle, H)$

$\uparrow \cong$

Solutions to $\{w_i(x)\}$

leads to study of logic over groups

If B is some compact top. space, then we have a 1-1
corresp

$$\text{Hom}(\pi_1(B), \text{Mod}(S)) / \sim \longleftrightarrow \{ \text{Isom classes of } S\text{-bundles over } B \}$$

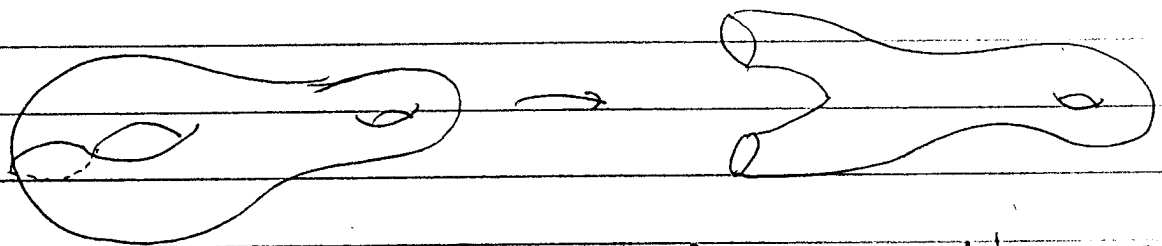
↑
postcomposition by inner-automorph

wher $S =$ compact oriented surface.

$$\begin{array}{ccc} S & \rightarrow & \mathbb{F} \\ & & \downarrow \\ & & B \end{array}$$

Theorem: S compact oriented surface, Γ f.g. group.
There are finitely many subgroups H_1, \dots, H_k
of $\text{Mod}(S)$ (which depend on Γ and S) s.t. for
 $\rho: \Gamma \rightarrow \text{Mod}(S)$ either

- (i) $\rho(\Gamma)$ is conjugate to H_i for some i
- or (ii) there is a finite index sgr $\Gamma_0 \leq \Gamma$
which has a non-trivial graph of groups
decomposition s.t. ρ takes the edge groups to
a subgroup of $\text{Mod}(S)$ fixing a curve.



"Example" Take $\Gamma = \pi_1(\text{torus}) = \langle a, b, c, d \mid [a, b][c, d] = 1 \rangle$
Map a, b anywhere in $\text{Mod}(S)$. Do this s.t.
 $[a, b]$ fixes a curve in S . Map c, d to the same place
as a & b (or anywhere else s.t. $[c, d]^{-1}$ goes to ρ)

$\langle h(a), h(b), h(c), h(d) \rangle \leq \text{Mod}(S)$ is the image of a surface group in $\text{Mod}(S)$.

Now for any k , $\langle h(a), h(b), T_y^k h(c) T_y^{-k}, T_y^k h(d) T_y^{-k} \rangle$ is a (hopefully different) image of $\Pi_1(S)$ in $\text{Mod}(S)$.

Def: $g \in \text{Mod}(S)$ is pseudo-Anosov if there is no pair (γ, k) , $k \in \mathbb{Z}$, γ s.c.c. on S s.t. $g^k(\gamma) = \gamma$.

There are free groups in $\text{Mod}(S)$ all of whose non-identity elements are pseudo-Anosov. "All pseudo-A"
However there are no known examples of one-ended all pseudo-Anosov subgroups.

Corollary: If Γ is one-ended, then there are at most finitely many conjugacy classes of all pseudo-Anosov subgroups of $\text{Mod}(S)$ which are isomorphic to Γ .

(In particular Γ a surface grp - previously proved by Bowditch)

Proof: Suppose infinitely many. There is an injective homomorphism $h: \Gamma \rightarrow \text{Mod}(S)$ (all pseudo-Anosov lying in case (ii)).

Then $\exists \Gamma_0 \leq \Gamma$ which admits a non-trivial graph of groups decomposition with edge groups mapping into curve stabilizers.

Thus Γ_0 splits over \mathbb{Z} $\Leftrightarrow \Gamma_0$ has infinitely many ends.

But then Γ has infinitely many ends.

qed

Corollary: Let Γ be a f.g. group with property (T). Then there are at most finitely many conjugacy classes of homomorphisms of Γ in $\text{Mod}(S)$.

One case of the proof of the theorem:

Case $S = T^2$ $\text{Mod } S = \text{SL}(2, \mathbb{Z})$. Take $\Gamma \rightarrow \text{SL}(2, \mathbb{Z})$
 $\forall f, i$
 $\Gamma_0 \rightarrow \text{free grp.}$