# Note Taker Checklist Form -*MSRI*

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**Talk Title and Workshop assigned to:**
Kleiner's proof of the polynomial growth theorem
Topics in geometric group theory

**Lecturer** (Full name): David Fisher

**Date & Time of Event:** 11/9/07 2:00 pm

**Check List:**

( ) Introduce yourself to the lecturer prior to lecture. Tell them that you will be the note taker, and that you will need to make copies of their own notes, if any.

( ) Obtain all presentation materials from lecturer (i.e. Power Point files, etc). This can be done either before the lecture is to begin or after the lecture; please make arrangements with the lecturer as to when you can do this.

( ) Take down all notes from media provided (blackboard, overhead, etc.)

( ) Gather all other lecture materials (i.e. Handouts, etc.)

( ) Scan all materials on PDF scanner in 2nd floor lab (assistance can be provided by Computing Staff) – Scan this sheet first, then materials. In the subject heading, enter the name of the speaker and date of their talk.

Please do **NOT** use **pencil** or colored pens other than black when taking notes as the scanner has a difficult time scanning pencil and other colors.

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**Please fill in the following after the lecture is done:**

1. List 6-12 lecture keywords:

2. Please summarize the lecture in 5 or less sentences.
Fisher explains Kleiner's new proof of Gromov's polynomial growth theorem. This ~~proof~~ ~~uses~~ approach involves a direct proof that the space of Lipschitz harmonic functions on a group of polynomial growth is finite ~~[illegible]~~ dimensional.

***Once the materials on check list above are gathered, please scan ALL materials and send to the Computing Department. Return this form to Larry Patague, Head of Computing (rm 214)***

# Kleiner's Proof of Polynomial growth theorem

David Fischer

<u>Thm</u> (Gromov) If a finitely generated group $G$ has polynomial growth, then $G$ has a finite index nilpotent subgroup.

Plan:
1. Reductions to a statement about harmonic analysis
2. Harmonic functions / maps
3. Two key inequalities
4. Prove the theorem

Gromov's strategy: Induction, enough to show $\exists\ G \twoheadrightarrow \mathbb{Z}$.
(Use Milnor-Wolf: polycyclic gp with poly. growth has finite index nilp. sub gp)

Enough to show $\exists$ $\infty$-image linear rep. of $G$.
$\Rightarrow G \twoheadrightarrow \mathbb{Z}$ using Tits' + Milnor-Wolf
(Shalom) ← amenable linear gps are virtually solvable

<u>Theorem</u>: (Mok, Korevaar-Schoen) If a group $G$ doesn't have property (T), then there is a ~~proper~~ fixed pt free isometric action of $G$ on a Hilbert space
And $\exists$ a ~~proper~~ unbounded $G$-equivariant Lipschitz (follows from equivariance) harmonic map from
$f : Cay(G, S) \longrightarrow \mathcal{H}$
(they talk about manifolds - "easy to adapt")

We'll use $G = Cay(G, S)$.

Observation: It is enough to see that $f(G)$ is contained in a fin. dim'l subspace. This gives a map $\rho : \Gamma \to O(m) \ltimes \mathbb{R}^n$ with $|\rho(G)| = \infty$.

<u>Fact</u> If $G$ has poly growth, then it does not have property (T).

To show $f(G)$ is contained in a fin. dim'l subspace, suffices to show that the space of Lipschitz harmonic functions on $G$ is fin. dim.

Why? $v \in \mathcal{H}$ $\quad L: \mathcal{H} \to \mathbb{R}$ $\quad L(v) = \langle \ , v \rangle$

then $L \circ f : G \to \mathbb{R}$ is Lipschitz + harmonic.

Theorem (Kleiner) The space of harmonic fcns ~~on G of~~ of polynomial growth on a group $G$ of polynomial growth is fin. dim.

Harmonic maps / functions: $\mathcal{G} = Cay(G, S) = (V, E)$ $\quad \underbrace{\begin{array}{l} g : V \to \mathbb{R} \text{ function.} \\ f : V \to \mathcal{H}. \end{array}}$

Def $g : V \to \mathbb{R}$ is harmonic if $\forall v \in V$ $\quad g(v) = \frac{1}{|S(v,1)|} \sum_{d(v,w)=1} g(w)$

value at $v$ is average of values of nearest neighbors.

$$\Leftrightarrow \sum_{d(v,w)=1} (g(v) - g(w)) = 0$$

Can define $f : V \to \mathcal{H}$ harmonic in the same way.

$f : V \to \mathcal{H}$ $\quad X = f(e)$. $\quad$ Left cayley graph: $sx$ are neighbors of $f(e)$.

$$E(f) = \sum_{s \in S} d^2(sx, x)$$

~~Exercise~~: Def2 $f$ is harmonic if it minimizes $E$ among equivariant maps.

Exercise Show def.s of $f$ harmonic are equivalent.

$\cdot$ Compute $DE$, min. where $DE = 0$ $\quad E$ is convex.

Using this def., it is clear that $L \circ f$ is harmonic.

Two inequalities $g : V \to \mathbb{R}$ $\quad |\nabla g| : E \to \mathbb{R}$

$$|\nabla g|(e) = |g(t(e)) - g(i(e))|$$

$$i(e) \overset{e}{\circ\!\!-\!\!-\!\!\bullet} t(e).$$

Poincaré inequality $f : B(3R) \to \mathbb{R}$, $B(3R)$ = some ball of radius $3R$.

$$\int_{B(R)} |f - f_R|^2 \leq C R^2 \frac{V(2R)}{V(R)} \int_{B(3R)} |\nabla f|^2$$

volume of ball of radius $s$ — $V(s) = |B(s)|$.

~~so just ball~~

$$f_R = \frac{1}{|B(R)|} \int_{B(R)} f$$

Doubling: $\dfrac{V(2R)}{V(R)} < D$ that doesn't depend on $R$.

- The above is true on any Cayley graph.

Reverse Poincaré Inequality $f : B(8R) \to \mathbb{R}$ harmonic

then $$R^2 \int_{B(R)} |\nabla f|^2 \leq C \int_{B(8R)} |f|^2$$

Will give a proof, fudging these inequalities so the domains of integration are the same on both sides.

( $B(R)$ instead of $B(8R)$, $B(3R)$ - "wildly false" but can "go back and fix it" - this is conceptually much easier.)

$V$ a f.d. space of Lipschitz harmonic functions on $G$.
$\dim V = k$.

$$Q_R(u,u) = \int_{B(R)} u^2 \quad \text{quadratic form on } V.$$

$Q_R$ is non-decreasing in $R$.

$$\liminf_{R\to\infty} \frac{V(R)\det(Q_R)^{1/k}}{R^d} < \infty \quad \text{for some } d.$$

Choosing scales (all scales large enough s.t. $Q_R > 0$.
~~R~~ Given any $w \in \mathbb{N}$, can choose $R_1, R_2$ s.t.
$\left(\frac{R_1}{R_2}\right)^2 \lesssim \frac{1}{e^w}$ and so the volume growth both
near $R_1$ and $R_2$ and between $R_1$ and $R_2$, "looks doubling".

$$\frac{V(kR_1)}{V(R_1)} < C(k)$$

Main application of choice of scales is to show $\exists$ a cover of $\mathcal{B} = \{B_i(R_1)\}$
of $B(R_2)$ s.t. 1) bounded # of overlaps
different centers
2) $|\mathcal{B}| \le C e^w = J$

("mimic Vitali covering lemmas)

$\Phi : V \to \mathbb{R}^J$, $\Phi_j(v) = \frac{1}{|B_j|}\int_{B_j} v$

Goal: show $\Phi$ is injective. (Shows $k < J$, finite)

Lemma: ~~u ∈ B(R₂)~~ then $Q_{R_2}(u,u) \le C V(R_1)|\Phi(u)|^2 + C R_1^2 \int_{B(R_2)} |\nabla u|^2$
$u: B(R_2) \to \mathbb{R}$
Pf: good cover & Poincaré inequality.

Given the lemma, apply reverse Poincaré ineq.

$$Q_{R_2}(u,u) \le CV(R_1)\,|\Phi(u)|^2 + C\left(\frac{R_1}{R_2}\right)^2 \int_{B(R_2)} u^2$$

$$Q_{R_2}(u,u) \le CV(R_1)\,|\Phi(u)|^2 + C\left(\frac{R_1}{R_2}\right) Q_{R_2}(u,u).$$

- On $\ker \Phi$, depends only on # generators

$$Q_{R_2}(u,u) \le C\left(\frac{R_1}{R_2}\right)^2 Q_{R_2}(u,u)$$

if $C\left(\frac{R_1}{R_2}\right)^2 < 1$, ~~then~~ (which we can choose values to get)

then $u = 0$.