On asymptotic dimension of Coxeter groups

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Outline of the talk

- Main result

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- Amalgamation theorem

Main Result

- AMALGAMATION THEOREM.
  $\operatorname{asdim}(A *_C B) \leq \max\{\operatorname{asdim} A, \operatorname{asdim} B, \operatorname{asdim} C + 1\}$.

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- THEOREM. *For Coxeter groups* (Γ, S)*,*

$$\operatorname{asdim} \Gamma \le \dim \Sigma(\Gamma, S) = \dim N(\Gamma, S) + 1$$

where $\Sigma(\Gamma, S)$ *is the Davis complex and* $N(\Gamma, S)$ *is the nerve.*

Coxeter groups

- A *Coxeter matrix* $(m_{uv})_{u,v \in S}$ is a symmetric matrix with $m_{vv} = 1$ and $m_{uv} \in \mathbb{N} \cup \{0\}$. A Coxeter matrix is *right-angled* if $m_{uv} \in \{0, 2\}$ for $u \neq v$.

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- A Coxeter matrix defines a *Coxeter group*

$$\Gamma = \langle S \ \mid (uv)^{m_{uv}}, u, v \in S\rangle$$

Thus $s^2 = 1\ \forall s \in S$.

Γ is generated by the set of reflections S.

Nereve and Davis Complex

- Nerve $N = N(\Gamma, S)$ of a Coxeter group Γ is a finite complex with the set of vertices S. Vertices $u_1, \dots, u_k$ span a simplex iff they generate a finite subgroup in Γ. Thus, u and v are joined by an edge iff $m_{uv} \neq 0$. In right-angled case fill all empty triangles, then 3-simplices, then 4-simplices and so on.

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- The Davis complex X is the image of a simplicial map $q : \Gamma \times Cone(\beta N) \to X$ define by the equivalence relation on vertices: $\gamma \times c_\sigma \sim \beta \times c_{\sigma'}$ iff $\sigma = \sigma'$ and $\gamma^{-1}\beta \in \Gamma_\sigma$ where $\sigma, \sigma' \subset N$ are simplices and c_σ, c_σ are their barycenters. $\Gamma_\sigma \subset \Gamma$ is generated by σ.

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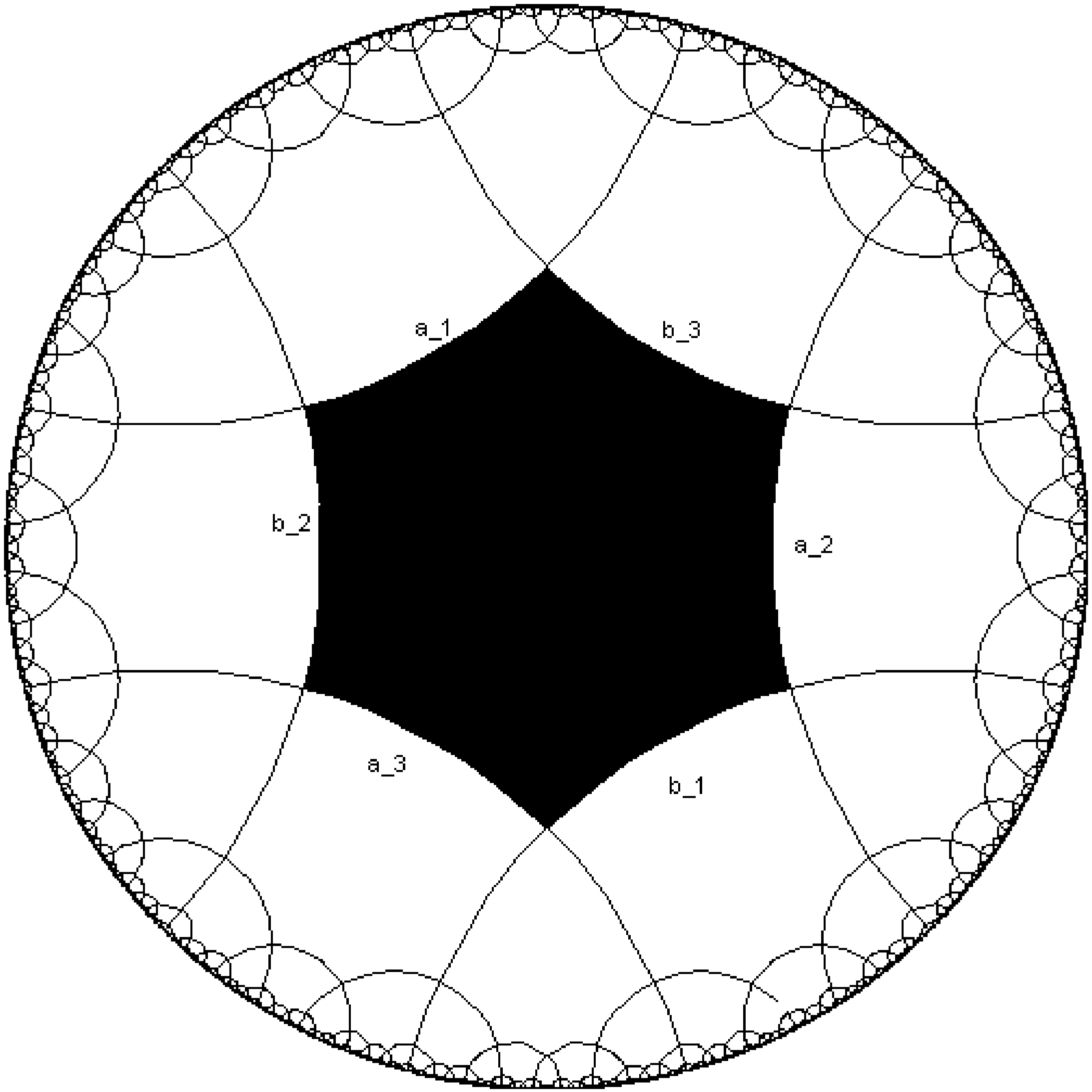
- A Coxeter group acts on its Davis complex by reflections.

Example



asdim **of Coxeter groups**

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- What is $\operatorname{asdim}\Gamma$ of Coxeter groups?

Asymptotic dimension

- DEFINITION.(Gromov) $asdim X \leq n$ *if for every uniformly bounded cover* $\mathcal{V}$ *of* X *there exists a uniformly bounded cover* $\mathcal{U}$ *of order* $ord\mathcal{U} \leq n+1$ *such that* $\mathcal{V} \prec \mathcal{U}$*.*

  Here $ord_x\mathcal{U} = |\{U \in \mathcal{U} \mid x \in U\}|$, and $ord\mathcal{U} = \max_{x \in X} ord_x\mathcal{U}$.

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- DEFINITION.(Lebesgue) $dimX \leq n$ *if for every open cover* $\mathcal{V}$ *of* X *there exists an open cover* $\mathcal{U}$ *of order* $\leq n + 1$ *such that* $\mathcal{U} \prec \mathcal{V}$.

Equivalent definition

- DEFINITION.(Gromov) $asdim X \leq n$ *if for every* $d < \infty$ *there exists a uniformly bounded cover* $\mathcal{U}$ *such that* $\mathcal{U} = \mathcal{U}^0 \cup \cdots \cup \mathcal{U}^n$ *and each family* $\mathcal{U}^i$ *is* d*-disjoint.*

  A family $\mathcal{A}$ of sets in a metric space X is d*-disjoint* if $dist(A, B) \geq d$ for all $A, B \in \mathcal{A}$, $A \neq B$ where $dist(A, B) = \inf\{dist(a, b) \mid a \in A, b \in B\}$.

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- DEFINITION.(Ostrand) $dim X \leq n$ *if for every* ϵ *there exists an* ϵ*-small open cover* $\mathcal{U}$ *such that* $\mathcal{U} = \mathcal{U}^0 \cup \cdots \cup \mathcal{U}^n$ *and each family* $\mathcal{U}^i$ *is disjoint.*

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- Since all word metrics d_S for finite S in a f.g. group Γ are quasi-isometric, we obtain that $asdim(\Gamma, d_S)$ for finitely generated group Γ does not depend on choice of the finite generating set S.

Asymptotic dimension

- THEOREM. [Gromov] $\operatorname{asdim} X \leq n$ *iff* $\forall \epsilon > 0$ *there is a uniformly cobounded* ϵ*-Lipschitz map* $f : X \to K$ *to a uniform* n*-dimensional simplicial complex.*
  Here
  uniform complex: $K \subset l_2(K^{(0)})$;
  uniformly cobounded map $f : X \to K$: $\exists C > 0$ such that $diam(f^{-1}(\Delta)) < C$ for all simplices $\Delta \subset K$.

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Dimension

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$L(\mathcal{U}) = \inf_{x \in X} \sup_{U \in \mathcal{U}} d(x, X \setminus U)$.

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- Roe's THEOREM. $HX^n(Y) = H_c(Y)$ *for uniformly contractible* Y*.*
  Y is *uniformly contractible* if $\forall R\, \exists S$ such that the inclusion $B_R(x) \subset B_S(x)$ is null-homotopic for all $x \in X$.

Estimate of asdim from below

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- PROPOSITION. $\operatorname{asdim}\Gamma \geq vcd\Gamma$ *for Coxeter groups* Γ*.*
- Proof. Let $vcd\Gamma = n$ and let Γ' be a torsion free finite index subgroup with $cd\Gamma' = n$.
  Then $H_c^n(\Sigma(\Gamma)) = H^n(B\Gamma', \mathbb{Z}\Gamma) \neq 0$.
  By Roe's Theorem $HX^n(\Sigma) \neq 0$.
  Since $H_c^n(\Sigma) = \check{H}^n(\alpha\Sigma)$, the group $H_c^n(\Sigma)$ is countable.
  Since $\lim^1$ cannot be countable (if $\neq 0$),
  $\lim_{\leftarrow} H_c^n(N_i) \neq 0$.
  Thus $\dim N_i \geq 0$. Hence $\operatorname{asdim}\Gamma \geq n$. $\square$

vcd of Coxeter groups

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  where CN is the cone over the nerve $N(\Gamma)$.
- *global cohomological dimension of a space*: $gcdK = \max\{n \mid H^n(K) \neq 0\}$.

  $Lk(\sigma, K)$ is the link of a simplex $\sigma \subset K$ = the union of all simplices σ' such that $\sigma' * \sigma \subset K$.

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  ’Yes’ for polycyclic groups (Bell-Dr.)

asdim **vs vcd**

- *Is always* $\operatorname{asdim}\Gamma = vcd(\Gamma)$?
- 'Yes' for hyperbolic groups (Buyalo-Lebedeva & Bestvina-Mess).
  'Yes' for polycyclic groups (Bell-Dr.)
- *Is* $\operatorname{asdim}\Gamma = vcd(\Gamma)$ *where* Γ *is a right-angled Coxeter group with the nerve an acyclic 2-complex?*
  Note that Γ is a candidate for a counterexamples to the Eilenberg-Ganea problem: $vcd\Gamma = 2$, $gd\Gamma = 3?$. It's unclear if $\operatorname{asdim}\Gamma$ should be with vcd.

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- Dr- Schroeder: $\operatorname{asdim}\Gamma \leq ch(N(\Gamma))$ where $ch(K)$ is the *chromatic number* of the complex (graph) K.

$ch(pentagon) = 3$ and $\operatorname{asdim}\Gamma = \dim(pentagon) + 1 = 2$.

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- THEOREM. $\operatorname{asdim}\Gamma \leq \dim N(\Gamma) + 1$ for right-angled Coxeter groups Γ.
- *Proof.* Induction on $\dim N(\Gamma)$ and induction on $|N(\Gamma)^{(0)}|$. If $\dim N = 0$, then Γ is virtually free and $\operatorname{asdim}\Gamma = 1$. Let $\dim N = n$. Then $|N^{(0)}| \geq n+1$. If $|N^{(0)}| = n+1$, then $N = \Delta^n$ is the n-simplex. Then Γ is finite an $\operatorname{asdim}\Gamma \leq 0$. If every vertex v of N is connected by an edge with any other vertex, then $N^{(1)} = \sigma^{(1)}$ for a simplex σ with $\dim\sigma > n$. Since Γ is right-angled $N = \sigma$. Contradiction with $\dim N = n$.

Estimate from above

- *Proof continued.* Thus, there is a vertex $v \in N$ such that $St(v, N)$ does not contain $N^{(0)}$. Consider $N_1 = St(v, N)$, $N_2 = N \setminus OSt(v, N)$, and $K = Lk(v, N)$. Then

$$\Gamma = \Gamma_{N_1} *_{\Gamma_K} \Gamma_{N_2}$$

where Γ_L is a subgroup of Γ generated by $L^{(0)} \subset S$. Note that $N(\Gamma_{N_i}) = N_i$ and $N(\Gamma_K) = K$. By induction $\operatorname{asdim} \Gamma_{N_i} \le n + 1$ and $\operatorname{asdim} \Gamma_K \le n$. By the Amalgamation Theorem,

$$\operatorname{asdim} \Gamma \le \max\{\operatorname{asdim} \Gamma_{N_i}, \operatorname{asdim} \Gamma_K + 1\} = n + 1.$$

$\square$

Action Theorem

- WEAK AMALGAMATION THEOREM. [Bell-Dr]

$$\operatorname{asdim} A *_C B \le \max\{\operatorname{asdim} A, \operatorname{asdim} B\} + 1.$$

The proof is based on the following

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- ACTION THEOREM. [Bell-Dr] Let Γ act on X by isometries and let $\operatorname{asdim} X \leq n$. Suppose that $\operatorname{asdim}(Stab_R(x_0)) \leq k \quad \forall R > 0$ where

$$Stab_R(x_0) = \{g \in \Gamma \mid d(g(x_0), x_0) \leq R\}$$

is the R-stabilizer of x_0 for some $x_0 \in X$. Then $\operatorname{asdim} \Gamma \leq n + k$.

Weak Amalgamation Theorem

- *Proof of the Weak Amalgamation Theorem.* $\Gamma = A *_C B$ acts on the Bass-Serre tree T whose vertices are the left cosets $\Gamma/A \coprod \Gamma/B$ and the vertices xA and xB, $x \in \Gamma$, and only them are joined by an edge. Note that $Stab_R(\{A\}) \subset (AB)^R$, $R \in \mathbb{N}$. The Action Theorem together with following Lemma complete the proof.

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Union Theorem

- The proof of the Lemma is based on the following: UNION THEOREM. *Let* $X = \cup_\alpha X_\alpha$ *be a metric space where the family* $\{X_\alpha\}$ *satisfies the inequality* $\operatorname{asdim} X_\alpha \le n$ *uniformly. Suppose further that for every* r *there is a* $Y_r \subset X$ *with* $\operatorname{asdim} Y_r \le n$ *so that* $d(X_\alpha \setminus Y_r, X_{\alpha'} \setminus Y_r) \ge r$ *whenever* $X_\alpha \ne X_{\alpha'}$. *Then* $\operatorname{asdim} X \le n$.

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- The family $\{X_\alpha\}$ of subsets of X satisfies the inequality $\operatorname{asdim} X_\alpha \leq n$ *uniformly* if for every $r < \infty$ one can find a constant R so that for every α there exist r-disjoint families $\mathcal{U}_\alpha^0, \dots, \mathcal{U}_\alpha^n$ of R-bounded subsets of X_α covering X_α.

Proof of Lemma

We prove that $\operatorname{asdim} AB\dots A(B) \le n$ by induction on the length of the product k. The inequality is a true statement for $k = 1$. Assume that it holds for k. Also assume that k is odd. Thus, $\operatorname{asdim} F_1 \dots F_k \le n$ where $F_{2i-1} = A$ and $F_{2i} = B$. We show that $\operatorname{asdim} F_1 \dots F_k B \le n$. Consider the family $\{wB \mid l(w) = k\}$. Since all sets wB are isometric to B, $\operatorname{asdim} wB \le k$ uniformly.
Given r we define $Y_r = AB \dots ACB_r$ where B_r is the r-ball in B. One can show that $d(wB \setminus Y_r, w'B \setminus Y_r) \ge r$ for $w \ne w'$. Then by the Union Theorem we obtain that $\operatorname{asdim}(F_1 \dots F_k \cap L_k)B \le n$ where L_k is the set of all elements $w \in A *_C B$ with $l(w) = k$. The inequality $\operatorname{asdim}(F_1 \dots F_m \cap L_{<m})B \le n$ follows from induction assumption and the Finite Union Theorem. □

Mapping Theorem

- The proof of the Action Theorem is based on the following:
  MAPPING THEOREM. *Let* $\pi : Y \to X$ *be Lipschitz and* $\forall R > 0$, $\operatorname{asdim} \pi^{-1}(B_R(x)) \leq k$ *uniformly on* k. *Then* $\operatorname{asdim} Y \leq \operatorname{asdim} X + k$.

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- The Mapping Theorem does not always give a good estimate.
  EXAMPLE: Let $\pi : \mathbb{H}^2 \setminus B \to S$ be the geodesic projection of the complement to the horoball onto the horosphere. Then by the Mapping Theorem $\operatorname{asdim} \mathbb{H}^2 \setminus B \leq 1 + 2$.

Asymptotic dimension revisited

- We say that $(r,d)-\dim X \leq n$ if for every $r > 0$ there exists a d-bounded cover $\mathcal{U}$ of X with $ord\mathcal{U} \leq n+1$ and with the Lebesgue number $L(\mathcal{U}) > r$. We refer to such a cover as to (r,d)-cover o X.

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- Clearly, $\operatorname{asdim} X \leq n$ if for every $r > 0$ there is d such that X admits an (r,d)-cover.

Partition Theorem

- PARTITION THEOREM. *Let X be a geodesic metric space. Suppose that for every $r > 0$ there is $d > 0$ and a partition $X = \cup_i^\infty W_i$ with $\operatorname{asdim} W_i \le n$ uniformly such that $(r,d)-\dim \cup_i \partial W_i \le n-1$ for all i. Then $\operatorname{asdim} X \le n$.*

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- In the above EXAMPLE the Partition Theorem gives right estimate $\operatorname{asdim}(\mathbb{H}^2 \setminus B) \leq 2$ by taking a regular partition $\{F_i\}$ of $S = \mathbb{R}$ and taking the preimages $W_i = \pi^{-1}(F_i)$.

Mapping Cylider Theorem

- *For every* $n \in N$ *there is a monotone tending to infinity function* $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ *with the following property: Given* $\epsilon > 0$, *let* $W = N_\lambda(Y) \cup N_\lambda(Z)$ *be the union of the* λ*-neighborhoods of* λ*-disjoint subests* Y *and* Z *in a geodesic metric space* X *with* $\lambda \geq 8/\epsilon$. *Then for every two covers* $\mathcal{V}$ *of* $N_\lambda(Z)$ *and* $\mathcal{U}$ *of* $N_\lambda(Y)$ *of the order* $\leq n+1$, *and with* $L(\mathcal{U}) > b(\mathcal{V}) > L(\mathcal{V}) \geq \mu(1/\epsilon)$, *there is a* $2b(\mathcal{U})$*-cobounded* ϵ*-Lipschitz map* $f : W \to M_g$ *to the mapping cylinder of a simplicial map* $g : Nerve(\mathcal{V}) \to Nerve(\mathcal{U})$ *between the nerves such that* $f|_Z = p_{\mathcal{V}}|_Z$ *where* $p_{\mathcal{V}} : N_\lambda(Z) \to Nerve(\mathcal{V}) \subset M_g$ *is the canonical projection.*

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- *The idea of the proof.* By means of the Mapping Cylinder Theorem an ϵ-Lipschitz uniformly cobounded map can be constructed to the n-complex K which is the union of the mapping cylinders of a simplicial maps induced by partial refinements $\mathcal{U} \prec \mathcal{W}_i$ of an (r,d)-cover of $\cup_i \partial W_i$ and $(2d, D)$-covers of W_i.

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- Let $n = \max\{\operatorname{asdim} A, \operatorname{asdim} B\}$. In view of the Weak Amalgamation Theorem we may assume that $\operatorname{asdim} C \leq n - 1$. Given $r > 0$ we construct a partition of the Cayley graph of Γ into G_is with $(r, d) - \dim \cup_i \partial G_i \leq n - 1$ and $\operatorname{asdim} G_i \leq n$. Then we apply the Partition Theorem.

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- We construct this partition using the action of Γ on the dual complex to the Bass-Serre tree.

The dual complex

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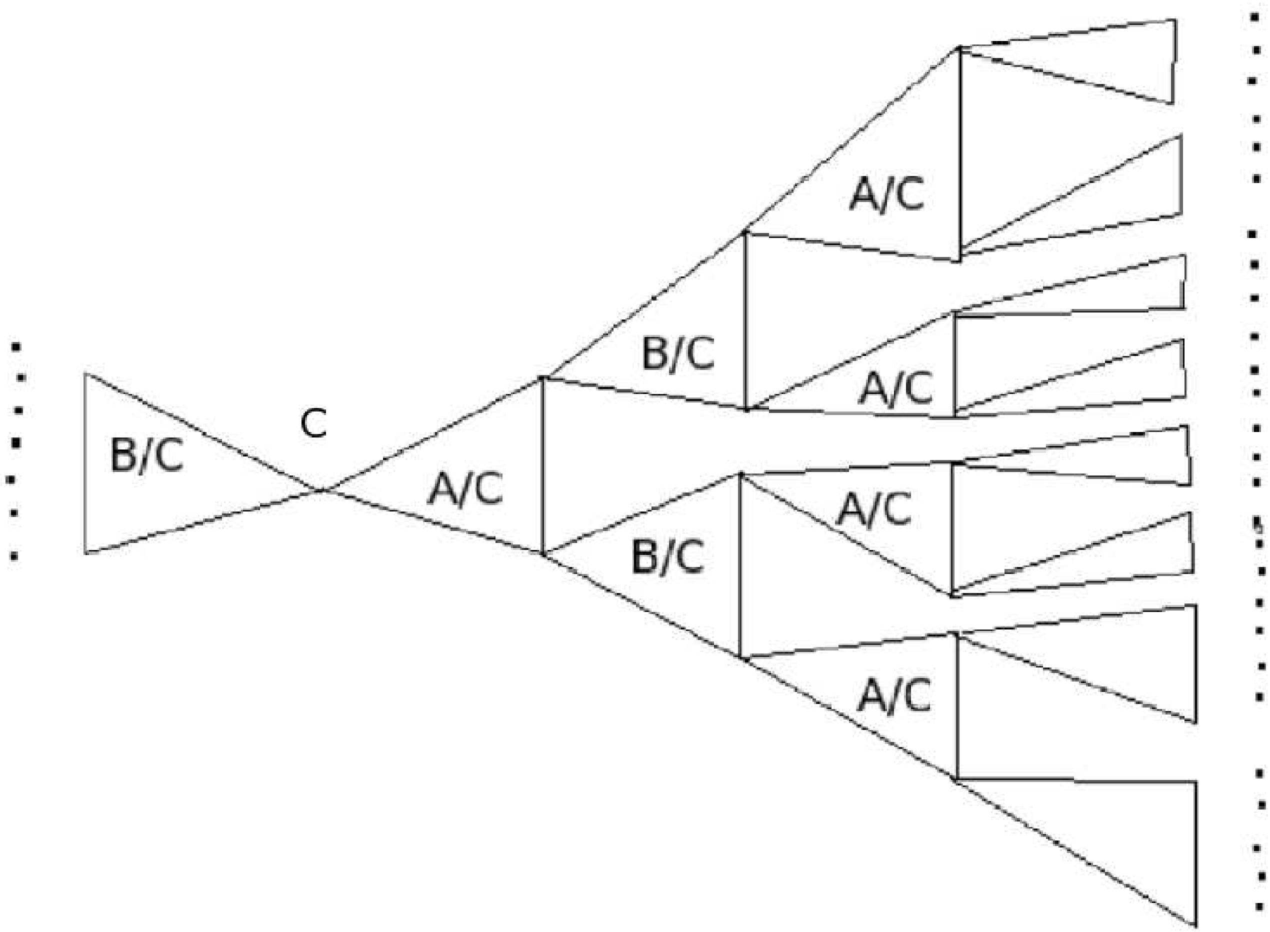
- The dual complex K serves better because the action of Γ on K is transitive and the projection to the orbit $\pi : \Gamma \to K$, $\pi(g) = g(x_0)$, is 1-Lipschitz. Hence π extends to a simplicial map of the Cayley graph.
  The corresponding projection for the Bass-Serre tree is 2-Lipschitz.

The Dual Complex



Amalgamation Theorem

- We consider a partition of $K = \cup F_i$ into pieces taken from the proof of $(R, S) - \operatorname{asdim} K \leq 1$ for sufficiently large R. Take the preimages $\pi^{-1}(F_i)$ and modify them into G_i, $i = 1, 2$.

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- **THE END**