Note Taker Checklist Form -*MSRI*

Name : Ioana Mihaila

E-mail Address/ Phone #: imihaila@csupomona.edu

Talk Title and Workshop assigned to:
Gromov's polynomial growth theorem

Lecturer (Full name): Indira Chatterji
Date & Time of Event: 8/24/07 1:30-2:30pm 4-5pm

Check List:

( ) Introduce yourself to the lecturer prior to lecture. Tell them that you will be the note taker, and that you will need to make copies of their own notes, if any.

( ) Obtain <u>all</u> presentation materials from lecturer (i.e. Power Point files, etc). This can be done either before the lecture is to begin or after the lecture; please make arrangements with the lecturer as to when you can do this.

( ) Take down <u>all</u> notes from media provided (blackboard, overhead, etc.)

( ) Gather <u>all</u> other lecture materials (i.e. Handouts, etc.)

( ) Scan all materials on PDF scanner in 2nd floor lab (assistance can be provided by Computing Staff) – Scan this sheet first, then materials. In the subject heading, enter the name of the speaker and date of their talk.

Please do **NOT** use **pencil** or colored pens other than black when taking notes as the scanner has a difficult time scanning pencil and other colors.

Please fill in the following after the lecture is done:

1. List 6-12 lecture keywords: ~~Gromov~~ growth polynomial nilpotence Gromov-Hausdorff convergence

2. Please summarize the lecture in 5 or less sentences.
Lecture sketched the proof of Gromov's theorem, which states that a finitely generated group of polynomial growth is virtually nilpotent.

Once the materials on check list above are gathered, please scan ALL materials and send to the Computing Department. Return this form to Larry Patague, Head of Computing (rm 214)

Indira Chatterji.

Theorem (Gromov 1981) If G is a finitely generated group of polynomial growth, then G is virtually nilpotent.

Plan: - basic definitions
- history
- limits of metric spaces

Let G finitely generated, S = generating set.
metric on G by $d(g,h) = \min_{n \in \mathbb{N}} \{ g^{-1}h = s_1 \dots s_n, s_i \in S \}$
$B(\gamma, n)$ = ball radius n centered at γ
$b_n = \# B(1,n)$

Def: G has polynomial growth if $\exists c, d$ s.t. $b_n \leq c n^d$ $\forall n$
- exponential growth if $b_n \geq a^n$ for some $a > 1$
- subexponential otherwise.
- intermediate if subexponential and not polynomial.

Def: If P is a property, a group G is "virtually P" if $\exists H \leq G$ of finite index s.t. H has property P.

Def: G, $H < G$, $[G,H] = \langle [g,h] = ghg^{-1}h^{-1} \mid g \in G, h \in H \rangle$
- Lower central series for G

$$G = G_0 > G_1 = [G, G_0] > G_2 = [G, G_1] > G_3 = [G, G_2] > \dots$$

- ~~Descending~~ Derived ~~normal~~ series of G.

$$G = G_0 > G^1 = [G, G_0] > [G^1, G^1] = G^2 > [G^2, G^2] \dots$$

Def: G is nilpotent if lower central series terminates in m steps

G is solvable if derived series terminates in m steps.

Nilpotent: $f: G \to G$ (fix $g \in G$)
$h \mapsto [g,h]$ $\quad f^n = 1$ some n

Exercise: If F is finite & N nilpotent, then
$1 \to F \to G \to N \to 1$ is virtually nilpotent

History: Milnor - Wolf (1968) If G is fin generated & solvable, then if G has polyn growth, G is virtually nilpotent.

Bass (1972) If G is virtually nilpotent, then it has polyn growth of degree $d = \sum_{i=0}^{c-1} i d_i$

d_i = free rank of A_i, $A_i = G_i / G_{i+1}$ (abelian group)

Tits (1972) → proved Tits alternative.

G finitely generated subgroup of a Lie group with finitely many components. Then either

(1) $G \geq \mathbb{F}_2$

or

(2) G is virtually solvable

(hence virtually nilpotent if of polynomial growth)

$c_1 n^d \leq b_n \leq c_2 n^d$

Pansu (1983) showed that

$\lim \frac{b_n}{n^d} = c$ if you have pol. growth

Ex: • $\mathbb{Z}^n$ is of polyn growth

• $\mathbb{F}_2$ is of exp. growth

Nilpotent ex.

Heisenberg $\begin{pmatrix} 1 & a & z \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$

$1 \to \mathbb{Z} \to H \to \mathbb{Z}^2 \to 1$

Solvable ex.

$\mathbb{Z}^2 \rtimes_{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}} \mathbb{Z}$

$\begin{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^n & \begin{matrix} a \\ b \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} < SL_3(\mathbb{Z})$

solvable but not nilpotent

Breuillard (2007) same for compactly gen groups.

(If G loc. compact, compactly generated group of polyn growth, then $\lim \frac{b_n}{n^d} = c$

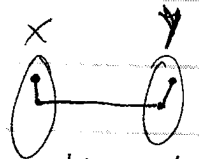
Limits of metric spaces

(Z, d) metric space, put a distance on the set of closed bdd. subsets of Z.

$$d_H(A,B) = \inf \{ \varepsilon > 0 \mid A \subseteq N_\varepsilon(B),\ B \subset N_\varepsilon(A) \}$$

Def: (X, d_X) (Y, d_Y) metric spaces, an admissible metric on $X \sqcup Y$ is a metric d s.t. $d|_X = d_X$ $d|_Y = d_Y$

Example



The Gromov-Hausdorff dist. between X, Y is given by
$$H(X,Y) = \inf \{ d_H(X,Y) \mid X \sqcup Y \subseteq (Z,d)\ d \text{ admissible metric} \}$$

Exercise: On the set of compact spaces, $H(X,Y) = 0$
$\iff$ X, Y are isometric.

For noncompact spaces H does not work. You have to chose basepts $x \in X$, $y \in Y$ define $\tilde{H}((X,x),(Y,y)) =$
$= \inf \{ \varepsilon > 0 \mid \exists d$ admissible on $X \cup Y$ s.t. $d(x,y) < \varepsilon$, $B(x, 1/\varepsilon) \subseteq N_\varepsilon(Y)$
$B(y, 1/\varepsilon) \subseteq N_\varepsilon(X) \}$

Def: Let (X_i, x_i) be a sequence of pointed metric spaces. Then it converges to (Y,y) if $\lim_{i \to \infty} \tilde{H}((X_i, x_i),(Y,y)) = 0$

Part II Plan

- limits of isometries
- limits of discrete groups
- sketch of proof.

Def: A family $\{B_j\}$ of compact metric spaces is <u>uniformly compact</u> if they have uniformly bdd. diameter and $\forall \varepsilon > 0$, $\exists N_\varepsilon$ s.t. each B_j can be covered by N_ε balls of radius ε.

Prop: Let (X_j, x_j) pointed metric spaces, proper (closed balls are compact). If $\forall r \geq 0$ the sequence $\{B(x_j, r) \subset X_j\}$ is unif compact, then $\exists$ subseq. of metric spaces $\{(X_j, x_j)\}$ converging to a proper metric space (Y, y).

Example: $X_1 = \mathbb{Z}$, $d(n,m) = |m-n|$ $\quad x_1 = 0$

$X_2 = \mathbb{Z}$ $\quad d(n,m) = \frac{1}{2}|m-n|$ $\quad x_2 = 0$

$\vdots$

$X_k = \mathbb{Z}$ $\quad d(n,m) = \frac{1}{k}|m-n|$ $\quad x_k = 0$

Theorem: For any polyn growth group G, any generating system S of G, $\exists$ sequence $r_1, r_2, \ldots$ $r_i \to \infty$ s.t. r-balls centered at 1 in G in the metric space $X_i = r_i^{-1} G$ is uniformly compact.

Remark: If $\lim (X_j, x_j) = (Y, y)$, by def $\exists$ metrics d_j on $X_j \cup Y$ s.t. $\forall r \geq 0$, $\forall \varepsilon > 0$, $d_j(x_j, y) \leq \varepsilon$, $B(y, r) \subseteq N_\varepsilon(X_j)$ and $B(x_j, r) \subseteq N_\varepsilon(Y)$.

We will talk about definite convergence when we choose that set of metrics.

We can make a sequence of pts $(x_1', x_2', \ldots) \in X_1 \times X_2 \times \ldots$ converge to a pt. in Y.

Def: A sequence of maps $f_j : X_j \to X_j$ converges to $f : Y \to Y$ if $\forall r \geq 0$, $\forall \varepsilon > 0$, $\exists \mu(\varepsilon, r), \exists N(\varepsilon, r)$ s.t. $\forall j \geq N$

$x' \in B(x_j, r)$, $y' \in B(y, r)$ s.t. if $d_j(x', y') \leq \mu$ ~~then~~ $d_j(f(x'), f(y')) \leq \varepsilon$.

Isometry lemma $(X_j, x_j) \to (Y, y)$ (def. convergence)
proper

and $f_j : X_j \to X_j$ isometries s.t. $d_j(x_j, f_j(x_j)) \leq C$
$\Rightarrow \exists$ subseq of f_j that converges to an isometry of Y.

Corollary: X_j are homogeneous $\Rightarrow$ so is Y.

Discrete groups: Use them to find sequence of $X_i = r_i^{-1} G$ Cayley graphs. Look at the limit (Y, y)

Properties:
(1) Y proper, locally compact
(2) Y is connected and locally connected
(3) Y is homogeneous, $L = \mathrm{Isom}(Y)$ is transitive
(4) Y is finite dimensional.

Remark: G acts on each X_i by isometries. Take $\gamma_i \in G$ s.t. $r_i^{-1} \|\gamma_i\| \leq C \quad \forall i$.
Then the seq of isometries $\{\gamma_i : X_i \to X_i\}$ will converge to an isometry of Y.
That gives a map $G \to \mathrm{Isom}(Y)$

Sketch of proof of Gromov's theorem

(1) Montgomery - Zippin (1955)

Let Y be a finite dimensional, locally compact, connected & locally connected metric space. Then if $L = \text{Isom}(Y)$ is transitive on Y, then L is a Lie group with finitely many components.

(2) Now we have $\varphi : G \to L$ homomorphism.

some sort of induction.

Lemma: a) If $0 \to K \to G \to \mathbb{Z} \to 0$ and if G is of pol. growth, then growth$(K) \leq$ growth$(G) - 1$

b) if ~~~~ K is virtually solvable, so is G.

Cases: 1. If $\text{Im}\,\varphi$ is infinite in L, express G as an extension $1 \to K \to G \to \mathbb{Z} \to 1$.

conclude using induction + Milnor - Wolf

2. $\text{Im}\,\varphi$ is finite, Look at $\ker \varphi \subseteq G$, which has finite index.

If $\ker \varphi$ abelian, done.

If not abelian, modify φ into $\varphi' : \ker\varphi \to L$, that has infinite image. Note: technical.

Gromov's thm on polynomial growth

<u>Thm</u> (Gromov 1981) If G is a finitely generat[ed] group of polynomial growth, then G is virtually nilpotent.

<u>Plan I</u>: • Explain the words
• History & 3 related question
• Limits of metric spaces
2 Very rough sketch of the proof

<u>Plan II</u>: 1. Limits of ch[...] gro[ups]

G, finitely generated, pick S a finite generating set, gives a metric on G by

$d(g,h) = \min\{n \in \mathbb{N} \mid g^{-1}h = s_1 \cdots s_n \;\; s_i \in S\}$

denote $B(\gamma, n) = \{g \in G \mid d(g,\gamma) \leq n\}$ the ball of radius n, and $b_n = \# B(1,n)$

<u>Def</u>: G has <u>polynomial growth</u> if there are constants s.t $b_n \leq C n^d$

<u>exponential growth</u> if for some $a > 1$ $b_n \geq a^n$, and <u>subexponential</u> otherwise.

<u>Intermediate growth</u> if subexponential non polynomial (first examples in 1984 by Grigorchuk)

+ rem for $\operatorname{gr} H \leq \operatorname{gr} G$ —)

Def: Given a property P, a group G is called "virtually P" if G contains a finite index subgroup that has P. (For instance, finite groups are virtually trivial).

Def: G a group, $H < G$, define

$[G,H] = \langle [g,h] = ghg^{-1}h^{-1} \mid g \in G, h \in H \rangle$

Lower central series of G

$G = G_0 > G_1 = [G,G] > G_2 = [G,G_1] > G_3 = [G,G_2] > \dots$

Descending normal ~~central~~ series of G

$G = G_0 > G' = [G,G] > [G',G'] = G^2 > [G^2,G^2] = G_3 > \dots$

Nilpotent: lower central series terminates as the trivial group in finitely many steps.

Soluble: upper central series

Remark: $A_i = G_i / G_{i+1}$ is abelian

nilpotent because $f: G \to G$, $x \mapsto [x,g]$

$\Rightarrow f^n = 0$ some n.

Exercise: $1 \to F \to G \to N \to 1 \Rightarrow G$ virt nilpotent.

Def: A Lie group is a group + structure of differentiable manifold + operations are smooth

History:

Bass (1972) Wolf: If G is virtually nilpotent, then it has polynomial growth (of degree $d = \sum_{i=0}^{s-1} i\, d_i$, $d_i = \operatorname{rank} A_i$)

Remark: Guivarc'h 1973 studied the locally compact Lie group case.

(same issue of Diff Geom)

Milnor-Wolf (1968, independently): If G is finitely generated solvable, and G of polynomial growth, then G is virtually nilpotent.

Tits (1972 Tits alternative) Let G be a finitely generated subgroup of a Lie group with finitely many components. Then either:

(a) $G > F_2$ ($\Rightarrow$ exp. growth)

(b) G is virtually solvable ($\Rightarrow$ virt. nilpotent if of polynomial growth).

Bass here $c_1 n^d \le b_n \le c_2 n^d$ d as above.

Pansu (1983) $\lim_{n\to\infty} \frac{b_n}{n^d} = c$

Breuillard (2007) Same for arbitrary loc. cpct grps.

Related questions:

A quasi-isometry between two metric spaces X, Y is a map $f: X \to Y$ s.t $\exists\, C$ a constant with

$$\frac{1}{C} d(x,x') - C \le d(f(x), f(x')) \le C d(x,x') + C$$
$$d(f(X), Y) \le C$$

Q: Your favorite class of groups $\mathcal{C}$ is it q.i. rigid? i.e for $G \in \mathcal{C}$, if a group H is q.i to G, then is $H \in \mathcal{C}$?

Note: For polynomial growth, that's easy to see, but for virtually nilpotent (or even abelian) you need Gromov's theorem.

Gromov's theorem allows to study algebraic properties (like nilpotency) using geometric tools only.

Sketch of proof: 3 key results.

(1) Montgomery - Zippin (1955): Let Y be a finite dimensional, locally compact, connected and locally connected metric space. If $L = \mathrm{Isom}(Y)$ is transitive on Y, then L is a Lie group with finitely many components.

(2) Tits alternative (+ Milnor - Wolf)

(3) Limits of metric spaces, limit of a discrete group (ancestor of asymptotic cones). For Γ of polynomial growth, that gives a Lie group L and a homomorphism $\varphi: \Gamma \to L$. Using Tits + polynomial growth we know that $\mathrm{Im}(\varphi)$ is virtually ~~nilpotent~~ solvable.

~~If Ker(φ) is abelian then Γ solvable so Milnor - Wolf shows that Γ in fact is nilpotent~~

~~If Ker(φ) finite, can reduce to abelian.~~

~~If Ker(φ)~~ 2 cases: ① $\mathrm{Im}(\varphi)$ infinite, express G as an extension $K \to G \to \mathbb{Z}$, induction says ~~K~~ K virt. nilpotent, so G virtually ~~solvable~~ virt. nilpotent hence

② $\mathrm{Im}(\varphi)$ is finite & $\mathrm{Ker}(\varphi) \underset{f.i}{<} G$, then either (a) $\mathrm{Ker}(\varphi)$ is abelian and done.

(b) $\mathrm{Ker}(\varphi)$ not abelian an you modify φ into a map $\varphi' : \Gamma \to L$ that has an infinite image. (that's some technical work here).

Remark: if $H < G$ of infinite index, then growth$(H) \leq$ growth$(G) - 1$ (recall that remark here)

Proceed by induction on the degree of polynomial and do the cases here

If $K \to G \to \mathbb{Z}$ & G of pol-growth, then

(1) $gr(K) \leq gr(G) - 1$

(2) If K virtually solvable, so is G.

In fact prove: $\exists \Delta \underset{f.i}{<} G$ ~~[illegible]~~ such that ~~[illegible]~~ $\varphi(\Delta) = \mathbb{Z} \subseteq L$

(Van den Dries & Wilkie)

& apply induction to $\ker \varphi$.

really here uses L Lie group, Jordan theorem.

<u>Limits of metric spaces</u>

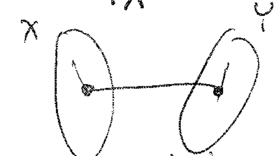
(Z, d) metric space, put a distance on the set of closed, bounded subsets in X (Hausdorff distance,

$$d_H(A,B) = \inf\{\varepsilon > 0 \mid A \subseteq N_\varepsilon(B),\ B \subseteq N_\varepsilon(A)\}$$

If look at it on all subsets, get something like a pseudodistance admitting infinite values.

Def: (X, d_X), (Y, d_Y) two metric spaces, an admissible metric on $Z = X \sqcup Y$ is a metric d such that $d|_X = d_X$ & $d|_Y = d_Y$

Ex: of admissible metrics



Def: Gromov-Hausdorff distance between two metric spaces given by

$$H(X,Y) = \inf\{d_H(X,Y) \mid X \sqcup Y \subseteq (Z,d) \text{ admissible}\}$$

EX: On the set of compact spaces, show that

$H(X,Y) = 0 \iff X, Y$ are isometric.

If X, Y noncompact, have to take basepoints, namely

Def: $\tilde{H}((X,x),(Y,y)) = \inf\{\varepsilon>0$ s.t $\exists\, d$ admissible on $X\cup Y$ s.t $d(x,y)<\varepsilon$, $B(x,1/\varepsilon)\subseteq N_\varepsilon(Y)$ and $B(y,1/\varepsilon)\subseteq N_\varepsilon(X)\}$

Remark: $\Delta \leq$ satisfied for small enough distances

Def: A sequence of pointed metric spaces (X_i,x_i) converges to (Y,y) if $\lim \tilde{H}((X_i,x_i),(Y,y))=0$

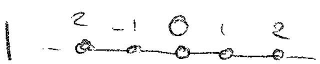
Remark: If X_i's compact with uniformly bounded diameter, that's $H(X_i,Y)\to 0$

Def: A family $\{B_j\}$ of compact metric spaces is <u>uniformly compact</u> if they have uniformly bounded diameter if $\forall \varepsilon>0\ \exists N=N(\varepsilon)$ such that each B_j can be covered by N balls of radius ε.

Prop: let (X_j,x_j) sequence of proper pointed metric space (ie closed balls are compact). If $\forall r\geq 0$, $\{B(x_j,r)\}$ is uniformly compact, then there is a subsequence converging to (Y,y) a proper metric space

Ex: • $\mathbb{Z}$, $d(n,m)=|n-m|$ $(X_\varphi, x_\varphi=0)$

$X_2 : \mathbb{Z}\ d(n,m)=\frac{1}{2}|n-m|$



$X_k : \mathbb{Z},\ d(n,m)=\frac{1}{k}|n-m|$

$\# B(x_k,r) = 1+2rk$

• $\mathbb{Z} = \langle \pm 1, \pm 100\rangle$ ...

Here the family of balls is uniformly compact, but in general it is not the case.

Theorem: For any polynomial growth group G and any finite generating system S of G, there is a sequence r_i such that $\lim r_i \to \infty$ and the family of r-balls centered at the identity in $r_i^{-1}G = X_i$ is uniformly compact

Remark: We talked about convergence, but if $\lim (X_i,x_i)=(Y,y)$, by definition $\exists$ metrics d_j on $X_j\cup Y$ such that $\forall r\geq 0$ $\forall \varepsilon>0$ $d_j(x_j,y)\leq\varepsilon$ $B(y,r)\subseteq N_\varepsilon(X_j)$ $B(x_j,r)\subseteq N_\varepsilon(Y)$

When we choose these metrics we speak about definite convergence

So can talk about a sequence of pts $x_j' \in X_j$ converging to same $y' \in Y$: $d_j(x_j', y') \to 0$ (in particular, $x_j \to y$).

Def : a sequence of maps $f_j : X_j \to X_j$ converges to $f : Y \to Y$ if $\forall r \geq 0$ $\forall \varepsilon > 0$ $\exists \mu = \mu(\varepsilon, r)$ & $N = N(r, \varepsilon)$ s.t $\forall j \geq N$ $(x', y') \in B(x_j, r) \times B(y, r)$ s.t $d_j(x', y') \leq \mu$ $\Rightarrow d_j(f_j(x'), f_j(y')) \leq \varepsilon$

Isometry lemma : $(X_j, x_j) \Rightarrow (Y, y)$ proper $f_j : X_j \to X_j$ isometries s.t $d_{X_j}(x_j, f_j(x_j)) \leq C$
$\Rightarrow$ subsequence on which $f_j \to f : Y \to Y$ isom

Corollary : X_j homogeneous $\Rightarrow$ Y also.

Discrete group :

Use thm to rescale (G, d_S) to make it converge to (Y, y)
$X_i = (r_i^{-1} G, e_i = e)$

(1) Y proper $\Rightarrow$ locally compact
(2) Y connected and locally connected
(3) (Cor above) Y is homogeneous hence $L = \mathrm{Isom}(Y)$ is transitive
(4) Y is f-dim (use unif. compactness)

G acts by isometries on X_i and if $\gamma \in G$ such that $r_i^{-1} \|\gamma\| \leq C < \infty$ then $\gamma = \gamma_j : X_j \to X_j$ sat. the isometry lemma $\forall j \geq i$ hence converge to $\ell_\gamma : Y \to Y$.
That gives a map $G \to \mathrm{Isom}(Y) = L$
this is obviously a homomorphism.

But if $\forall j$ take $\gamma_i \in \Gamma$ such that $r_i^{-1} \|\gamma_i\| \leq$ then $\{\gamma_i\}$ too converges to an isometry $Y \to Y$.

Kernel of this map studied using :

$$D(\gamma, r) = \sup_\beta \frac{\mathrm{dist}(\gamma\beta, \beta)}{\mathrm{length}(\beta^{-1}\gamma\beta)} \qquad \begin{array}{l} \gamma \in \Gamma \\ r \in [0, \infty) \\ \beta \in B(e, r) \end{array}$$

$$D(\Gamma, r) = \sup_{\substack{\text{gen set} \\ \text{for } \Gamma}} D(\gamma, r)$$

if $D(\Gamma, r)$ bounded, then ker abelian.
otherwise $r_i^{-1} D(\Gamma, r_i)$ same $r_i \to \infty$
find sequence α_i s.t $r_i^{-1} D(\alpha_i^{-1} \Gamma \alpha_i, r_i) = \varepsilon$
those α_i will modify the map $\Gamma \to L = \mathrm{Isom}(Y)$

Plan I: basic def's + nilp. facts.
history
G-H limits

Plan II: limits of isom
discrete grps
sketch of proof
further directions

Ex solvable $\mathbb{Z}^2 \rtimes_{\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)} \mathbb{Z}$ $\left(\begin{matrix}\begin{pmatrix}2&1\\1&1\end{pmatrix}^n & \begin{matrix}a\\b\end{matrix}\\ \begin{matrix}0&0\end{matrix} & 1\end{matrix}\right)$

Heis. $\begin{pmatrix}1&a&c\\0&1&b\\0&0&1\end{pmatrix}$ $\mathbb{Z} \to H \to \mathbb{Z}^2$