Median Spaces and Applications

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Based on joint work with C. Druku and F. Haglund
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Abstract

Median spaces are metric spaces in which given any 3 points, there exists a unique median point, that is a point lying on 3 geodesics between those 3 points. We will explain how median spaces are a natural generalization of CAT(0) cube complexes, how one characterizes property (T) and the Haagerup property using median spaces. This is joint work with C. Drutu and F. Haglund.

Let (X, d) be a metric space. For $x, y \in X$ say that $z \in X$ is between x and y ($z \in I(x, y)$) if $d(x, z) + d(z, y) = d(x, y)$. Call X a median space if $\forall x, y, z$ there exists a unique $t \in I(x, y) \cap I(y, z) \cap I(z, x)$, where $t = m(x, y, z)$ is called a median point.

The plan for the talk will be

1. Motivation and Results
2. Median spaces and measured wall spaces
3. Ideas of proof

Main Motivation

Theorem 1. *Let G be a locally compact, second countable group, then G has property (T) if and only if any isometric action on a median space has a bounded orbit.*

G is aTmenable if and only if there exists a proper action of G on a Median Space. The "if" direction of the proof is due to B. Nica (2002) which has two analogues. G is FH if replace Median Space by a Hilbert Space, and G is FA if replace Median Space by a tree.

Niblo-Roller (T) implies there is a fix points action on a CAT(0) cube complex.

Cherix-Martin-Valette (2003) if replace median by measured wall space.

Bader-Furman-Gelander-Monod (2006) if replace median by $L^p(M)$ for $0 < p \leq 2$.

Fisher-Margulis (2006) $\exists \epsilon > 0$ $L^p(M)$ for $2 \leq p < 2 + \epsilon$.

Pansu (1995) $G = Sp(n, 1)$ fix point free actions on $L^p(G)$ for $p > 4n + 2$.

Cornulier-Tessera-Valette (2007) even proper actions.

Bourdon-Pajot (2003) for all non-elementary hyperbolic groups can find fix point free action on $\ell^p(G)$ for p big enough.

Yu (2005) G hyperbolic implies a proper action on $\ell^p(G \times G)$ for p large enough.

Some partial converses are the following

Delorme-Guichardet FH=(T)

Martiu-Cheix-Valette If G is countable and has a bounded orbit action on a measured wall space implies property (T).

Robertson-Steger (1998) G countable and every measure definite kernel is bounded implies G has property (T).

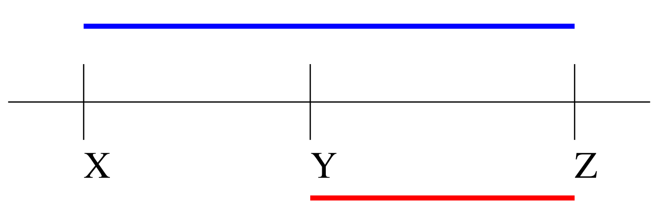
Bader-Furman-Gelander-Monod If there exists $p > 1$ such that every action on a $L^p(\Gamma)$ has fixed points implies that G has property (T).

Corollary 2. *If $p > 0$ and has bounded orbits implies that G has property (T).*

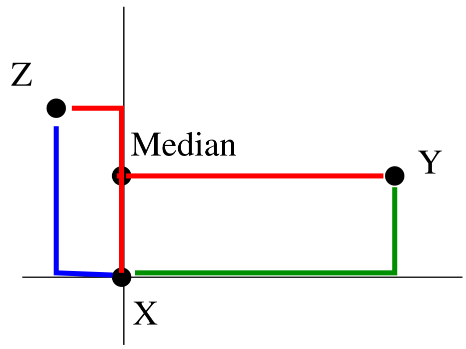
Definition 3. *G is aTmenable if it admits a proper action on a Hilbert Space.*

Theorem 4.
- *Any space with measured walls embeds isometrically in a median space.*
- *Any median space can be given a structure of a measured wall space.*
- *Any median space isometrically embeds in $L^1(W, \Gamma)$.*

Example 5. *1. $\mathbb{R}$ with the metric $d(x, y) = |x - y|$*

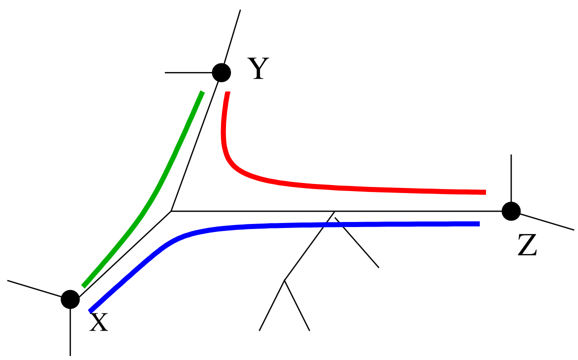


2. $\mathbb{R}^2$ *with the metric* $d\left(\left(\begin{array}{c} x_1 \\ x_2 \end{array}\right), \left(\begin{array}{c} y_1 \\ y_2 \end{array}\right)\right)$.



In each case you see the following picture which shows that the median $m(x, y, z) = y$.

Other examples are $\mathbb{R}$*-trees, and products of* $\mathbb{R}$*-trees. A generalization of the prod-*



ucts of $\mathbb{R}$*-trees is the fact that a product* $(X_1 \times X_2, d_1 + d_2)$ *of median spaces is again a median space.*

Definition 6. *A median graph is a graph such that the* 0*-skeleton with the graph metric is a median space (for the graph metric all edges have length one).*

Chepoi 2006 Any median graph is the 1-skeleton of a $CAT(0)$ cube complex.

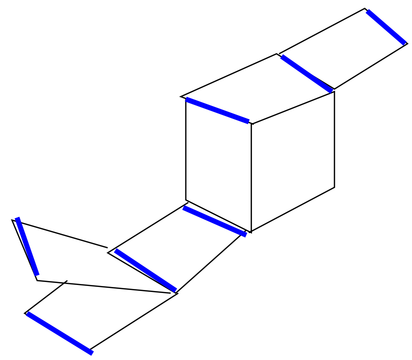
Roller 1998 1-skeleton's of $CAT(0)$ cube complexes are median graphs.

Definition 7. *A space with measured walls* (X, ω, B, μ) *where* X *is a set,* $\omega = \{(h, h^c = x - h)\ h \subset X\}$*,* B *is a* σ*-algebra of subsets of* ω*,* μ *is a measure on* B *such that* $\mu(\omega(x|y)) = \mu\{(h, h') \in \omega | x \in h, y \in h')\} < \infty\ \forall x, y$.

Example 8. 1. *A tree.*

2. $CAT(0)$ *cube complexes have a discrete wall space structure.*

   Converse is true also. Out of a wall space build a $CAT(0)$ *cube complex. In this way it seems that median spaces are a natural generalization of* $CAT(0)$ *cube complexes.*



Important Example

Let $(X, \mathcal{B}, \mu)$ be a measure space and $A \subset \mathcal{B}$.

$$\mathcal{B}_A = \{B \subset X \text{ such that } A \cap B \in \mathcal{B} \text{ and } \mu(A \cap B) < \infty\}$$

and define the pseudo-distance $\rho d_\mu(B, C) = \mu(B \cap C)$.

Claim 9. *$(\mathcal{B}_A, d_\mu)$ is a median space.*

$$I(B, C) = \{B \cap C \subset B \subset B \cup C\}$$

$$m(B, C, D) = (B \cup C) \cap (C \cup D) \cap (D \cup B)$$

Note that $\hat{\mathcal{B}_A} \hookrightarrow L^1(X, \mu) \cap L^2(X, \mu)$. Now $L^1(X, \mu)$ is a median space. If $f, g, h \in C(X, \mathbb{R}) \cap L^1(X, \mu)$ then $m(f, g, h)(x) = m(f(x), g(x), h(x))$.

Definition 10. *Median algebra is a set X with $I : X \times X \to \mathcal{P}(X)$ such that*

- $I(x, x) = \{x\}$
- *$z \in I(x, y)$ then $I(x, z) \subset I(x, y)$*
- $|I(x, y) \cap I(y, z) \cap I(z, x)| = 1$

Remark 1. *If X is a metric (median) space and $I(x, y) = \{z \in X \text{ between } x \text{ and } y\}$ then X is a median algebra.*

Idea of Proof of Main Theorem

We start with a proposition that could also be taken as the definition of property (T).

Definition 11. *(Delorme-Guichardet) G is a locally compact second countable group with property (T) if and only if any G-invariant conditionally negatively definite kernel is bounded.*

The second part of the if and only if statement means that if $\varphi : X \times X \to \mathbb{R}_+$ $\forall n \in \mathbb{N}$, $\zeta_1, \ldots, \zeta_n \in \mathbb{R}$ such that $\sum_{i=1} \zeta_i = 0$ and $x_1, \ldots, x_n \in X$ then

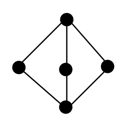
$$\sum_i \sum_j \zeta_i \zeta_j \varphi(x_i, x_j) \leq 0$$

Want to look at median definite kernels $\varphi : X \times X \to \mathbb{R}_+$ that factor through $Y \times Y$ such that there exists $f : X \to Y$ (X, Y median spaces) such that $\varphi(x, y) = d(f(x), f(y))$.

Robertson-Steger Measure definite kernels $\sim$ conditionally negatively definite kernels.

We Show Measure definite kernels $\sim$ median definite kernels.

One last remark is that



cannot embed in a median space.

Exercise 12.

$$(\mathbb{R}^2, euclidean) \hookrightarrow L^1([0,1])$$

given by

$$(a, b) \mapsto a\cos(t) + b\sin(t)$$

is an isometric embedding.