Note Taker Checklist Form -*MSRI*

Name : Ioana Mihaila

E-mail Address/ Phone #: imihaila@csupomona.edu

Talk Title and Workshop assigned to: Coxeter Groups & Artin Groups I / II
Intro to Geometric Group Theory

Lecturer (Full name): Ruth Charney

Date & Time of Event: 8/27/07 10:45 – 11:45 am
8/29/07 9 – 10 am

Check List:

( ) Introduce yourself to the lecturer prior to lecture. Tell them that you will be the note taker, and that you will need to make copies of their own notes, if any.

( ) Obtain all presentation materials from lecturer (i.e. Power Point files, etc). This can be done either before the lecture is to begin or after the lecture; please make arrangements with the lecturer as to when you can do this.

( ) Take down all notes from media provided (blackboard, overhead, etc.)

( ) Gather all other lecture materials (i.e. Handouts, etc.)

( ) Scan all materials on PDF scanner in 2nd floor lab (assistance can be provided by Computing Staff) – Scan this sheet first, then materials. In the subject heading, enter the name of the speaker and date of their talk.

Please do **NOT** use **pencil** or colored pens other than black when taking notes as the scanner has a difficult time scanning pencil and other colors.

Please fill in the following after the lecture is done:

1. List 6-12 lecture keywords: Coxeter Group, Artin Group, Deligne Complex, Salvetti Complex

2. Please summarize the lecture in 5 or less sentences.
The lecture defined Coxeter & Artin Groups, gave examples, then defined the associated complexes W_Γ & A_Γ. Part II talked about complexes associated to Coxeter & Artin groups, defining Davis, Deligne & Salvetti complexes, listing properties and some open questions.

Once the materials on check list above are gathered, please scan ALL materials and send to the Computing Department. Return this form to Larry Patague, Head of Computing (rm 214)

I Definitions & Examples

Coxeter graph

Γ = finite, simplicial, labelled graph

vertices : $S = \{s_1, \ldots, s_n\}$

edges : $\overset{m_{ij}}{\underset{s_i \quad s_j}{\circ\!\!-\!\!-\!\!-\!\!\circ}}$ $\quad m_{ij} \geq 2$

(convention : $m_{ij} = \infty$ if no edge)

Coxeter group

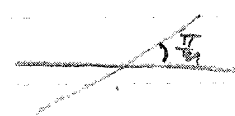
$$W_\Gamma = \langle S \mid s_i^2 = 1,\ (s_i s_j)^{m_{ij}} = 1 \rangle$$

= discrete linear gp generated by reflections

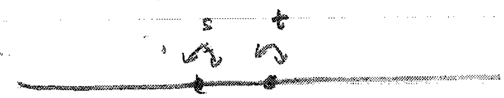
reflection (w.r.t $v \in \mathbb{R}^n$, $\|v\|_B = 1$) :

$r_v(w) = w - 2B(v,w)v$ $\qquad B$ = bilinear form

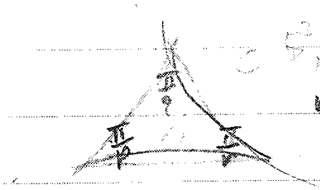
Eg 1) $n=2$: $W = \langle s, t \mid s^2 = t^2 = 1, (st)^m = 1 \rangle = D_{2m}$



if $m = \infty$: $W = \mathbb{Z}_2 * \mathbb{Z}_2 = D_\infty$

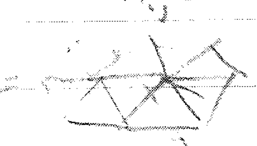


2) $n=3$: $W = T(\underbrace{p,q,r}_{m_{ij}\text{'s}}) =$



$\subset S^2, \mathbb{R}^2, \mathbb{H}^2$ reflect across walls of triangle

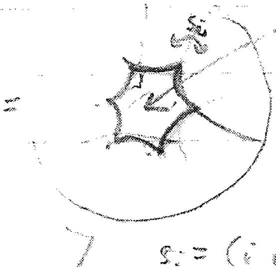
$T(3,3,3) =$



3) $n \geq 5$: $W = \langle s_1, \ldots, s_n \mid s_i s_{i+1} = s_{i+1} s_i \rangle$ (adjacent walls)

"right-angled" Coxeter groups

$\mathbb{H}^2 =$



right-angled n-gon

4) $n > 2$ $\quad W = \Sigma_n$ = sym. gp on n letters $= \langle s_1, \ldots, s_{n-1} \mid$ $\qquad$ $\rangle$ $\quad s_i = (i\ i{+}1)$

<u>Remark</u>: can rewrite $(s_is_j)^{m_{ij}}=1$ as $s_is_js_i\cdots = s_js_is_j\cdots$

<u>Artin group</u>

$$A_\Gamma = \langle S \mid \underbrace{s_is_js_i\cdots}_{m_{ij}} = \underbrace{s_js_is_j\cdots}_{m_{ij}} \rangle$$

$$= \pi_1\begin{pmatrix}\text{hyperplane complement} \\ \text{associated to } W_\Gamma\end{pmatrix}$$

$W \curvearrowright \mathbb{R}^n$ as reflection gp $\rightsquigarrow W \curvearrowright \mathbb{C}^n$

$Y =$ non-singular pts of this action

$$= \mathbb{C}^n - \bigcup_{\substack{r\in W \\ \text{reflection}}} H_r \quad \text{fixed by } r$$

$Y \longrightarrow Y/W$ covering space

<u>Thm</u>: (Brieskorn '71) $\pi_1(Y/W) = A$

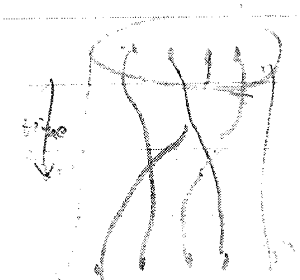
<u>Eg</u>: $W = \Sigma_n =$ symm gp on n letters $\curvearrowright \mathbb{R}^n \rightsquigarrow W \curvearrowright \mathbb{C}^n$

reflection hyperplanes: $H_{ij} = \{(z_1,\dots,z_n) \mid z_i = z_j\}$

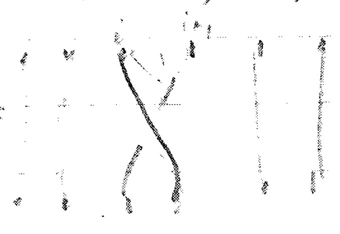
$Y/W = \mathbb{C}^n - \bigcup H_{ij}/W =$ config space of n distinct (unordered) pts in $\mathbb{C}$

$\pi_1(Y/W) =$ braid gp on n strands

$= \langle s_1,\dots,s_n \mid s_is_j = s_js_i, \dots \rangle$



$s_i =$



Remark: All Artin gps are infinite, in fact they are conjectured to be torsion-free.

Eg: $\Gamma = \underset{s}{\bullet}\overset{4}{\text{———}}\underset{t}{\bullet}$ $\quad \overset{W = D_8}{A} = \langle s, t \mid \underbrace{stst}_{\Delta} = tsts \rangle$ is big!

Δ is central: $t\underbrace{(stst)}_{\Delta} = \underbrace{(tsts)}_{\Delta}t$, $\text{Center}(A) = \langle \Delta \rangle$

$A / \langle \Delta \rangle = \langle \overset{st}{\overset{\shortparallel}{x}}, \overset{s}{\overset{\shortparallel}{y}} \mid x^2 = 1 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}$ $\qquad tsts = s^{-1}(\overset{x^2}{stst})s = y^{-1}x^2 y$

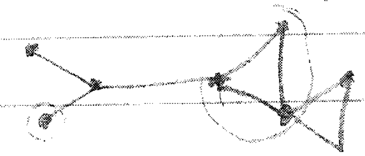
Def: A_Γ is finite type (or spherical) if W_Γ is finite

Notation: $T \subseteq S$, W_T, A_T

II. Geometry: complexes associated to W_Γ, A_Γ

For $T \subseteq S$,

W_T = subgp of W gen by T = Coxeter gp assoc to subgraph of Γ spanned by T

Γ = 

A_T = subgp of A gen by T =

Davis Complex for W (Tits, Davis)

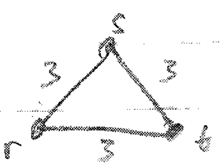
$\mathcal{D}_W = |\{wW_T \mid T \subseteq S,\ W_T \text{ finite}\}|$

$W \curvearrowright \mathcal{D}_W$ left mult, proper, cocompact

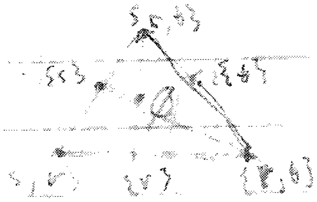
stab$(wW_T) = wW_Tw^{-1}$ finite!

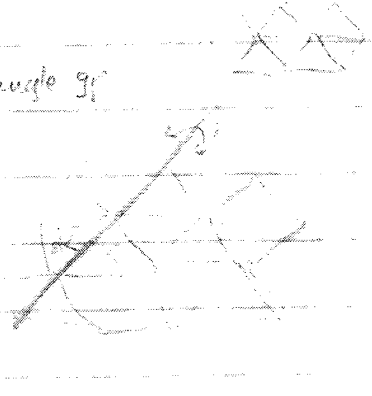
$K = |\{W_T \mid T \subseteq S,\ W_T \text{ finite}\}|$ fund domain

$\mathcal{D}_W = W \times K/\sim$

Eg: Γ = 

W_Γ = Euclidean triangle gp

K: 

$\mathcal{D}_W$ = 

Remark: generators s_i act as "reflections" on $\mathcal{D}_W$; fixed set of reflection called "wall"

What's $\mathcal{D}_W$ good for?

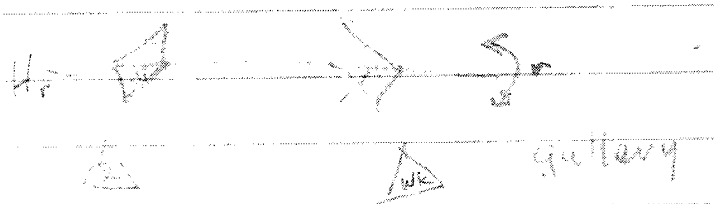
1) Dictionary:

combinatorial prop of W $\longleftrightarrow$ geometric prop of $\mathcal{D}_W$

$w = s_{i_1} \cdots s_{i_k}$ word — "gallery" in $\mathcal{D}_W$ from K to wK

minimal length word — gallery crosses each wall at most once

non-minimal $\Longrightarrow w = s_{i_1} \cdots \hat{\ } \cdots \hat{\ } \cdots s_{i_k}$ "Exchange Condition" — non-minimal $\Longrightarrow$ can shorten by reflecting



2) nice geometry:

Davis: $\mathcal{D}_W$ contractible

Moussong: $\mathcal{D}_W$ CAT(0) $\Longrightarrow$ W is CAT(0) gp

$\mathcal{D}_W$ hyperbolic $\Longleftrightarrow$ W has no $\mathbb{Z}^2$ subgps $\Updownarrow$ W is word hyperbolic

3) Interesting spaces in their own right.

Deligne complex for A (Deligne, C-Davis, v.d.Lek)

$$\mathcal{D}_A = \big|\{aA_T \mid T \subseteq S,\ A_T \underset{\substack{\Vert\,\text{def} \\ W_T \text{ finite}}}{\text{finite type}}\}\big|$$

$$= A \times K/\sim$$

$A \curvearrowright \mathcal{D}_A$, stabilizers aA_Ta^{-1} are not finite
cocompact but not proper!
$\mathcal{D}_A$ not locally finite!

Salvetti complex for A (Salvetti)

$A \underset{\sigma}{\twoheadrightarrow} W$ has a set theoretic section:
$w \in W$, write a shortest word $w = s_{i_1}s_{i_2}\cdots s_{i_k}$
set $\sigma(w) = s_{i_1}s_{i_2}\cdots s_{i_k} \in A$

Write $\widehat{W} = \sigma(W)$, $\widehat{W}_T = \sigma(W_T)$

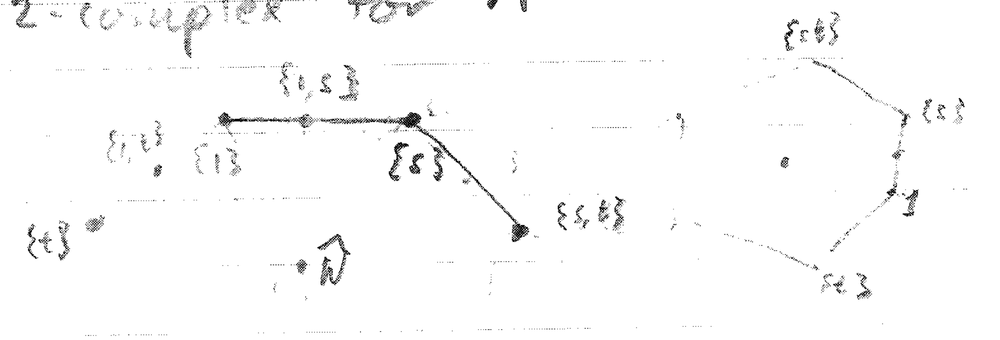
$$\mathcal{S}_A = \big|\{\underset{\text{set}}{a\widehat{W}_T} \mid T \subseteq S,\ W_T \text{ finite}\}\big|$$

$A \curvearrowright \mathcal{S}_A$ cocompact, free!

Eg: Γ: $\underset{s}{\bullet} \overset{m}{\text{———}} \underset{t}{\bullet}$ $\quad W = D_{2m}$

exercise: $\mathcal{S}_A = {}_{\wedge}^{\text{bary-subdiv}}$ Cayley 2-complex for A

$a\widehat{W}_\emptyset = \{a\}$



Topology of $\mathcal{D}_A$, $\mathcal{S}_A$

Fact: $\mathcal{D}_A \underset{\text{Deligne}}{\simeq} \tilde{Y} \underset{\text{Salvetti}}{\simeq} \mathcal{S}_A$ $\qquad$ $\tilde{Y}$ = univ cover of hyp comp

$\tilde{Y} \to Y \xrightarrow{\text{cov}} Y/W$

Conj. $\tilde{Y}$ is contractible

Conj $\Leftrightarrow$ $\tilde{Y}/A \overset{\simeq}{=} Y/W$ is a $K(A,1)$-space

$\Leftrightarrow$ $\mathcal{S}_A/A$ is a (finite) $K(A,1)$-space

$\Rightarrow$ A type F, torsion-free, ...

- (Deligne) conj true for finite type A.
- (C-Davis) two metrics on $\mathcal{D}_A$
  - Moussong metric : CAT(0) if $\mathcal{D}_A$ 2-dim'l (some 3-dim)
  - cubical metric : CAT(0) $\Leftrightarrow$ A is "FC-type"
- (C+Crisp) : suff cond for CAT(-1) metric $\Rightarrow$ A (weakly) rel hyp

Techniques + Open Questions

1. A *finite type* ($\Leftrightarrow W$ finite)

Powerful combinatorial techniques : Garside structure
(Garside, Deligne, Birman-Ko-Lee, Brady, Bessis)

A^+ = monoid of positive words

partial order on A^+ : $a \le b$ if $\exists c \in A^+$, $ac = b$.

Key facts

- A^+ is a lattice w.r.t. $\le$ ($\exists$ l.u.b's and g.l.b's)
- $\exists\, \Delta \in A^+$ s.t. $A = A^+[\Delta^{-1}]$

Let $M = \{a \in A^+ \mid a \le \Delta\} \supset S$

Garside structure $\rightsquigarrow$ canonical forms for A as words in M

$$a \in A^+ : \quad m_1 = g.l.b(a, \Delta) \Rightarrow a = m_1 a_1$$
$$m_2 = g.l.b(a_1, \Delta) \Rightarrow a = m_1 m_2 a_2$$
$$\vdots$$
$$a = m_1 m_2 \cdots m_k \quad \text{canonical form}$$

$$g \in A^+ \Rightarrow g = a\Delta^{-n}, \; a \in A^+, \; n \text{ minimal}$$
$$\Rightarrow g = m_1 \cdots m_k \Delta^{-n} \quad \text{canonical form.}$$

Garside structure

$\Rightarrow$ canonical forms for elts. of A

$\Rightarrow$ biautomatic structure on A

$\overset{\text{Deligne}}{\Rightarrow}$ contractible

$\Rightarrow \mathcal{S}_A/A$ finite $K(A,1)$.

2. Is A a CAT(0) group?

(2) A ~~affine~~ type (Euclidean)

For $\tilde{A}_n$, $\tilde{C}_n$, $\tilde{B}_n$, conj true.
For $\tilde{A}_n$, $\tilde{C}_n$ $A <_{\text{finite index}}$ Mod(punctured sphere)

Q: (McCammond) Is there an analogue of a Garside structure ~~for~~ affine type A?

(3) Large type ($\subseteq$ 2-dim'l)
all $m_{ij} \geq 3$ $\Longrightarrow$ $\mathcal{D}_A$ 2-dim'l

combinatorial techniques ~ small techniques
Appel-Schupp, Peifer, Juhász
$m_{ij} \geq 4 \Rightarrow$ biautomatic

geometric techniques: $\mathcal{D}_A$ CAT(0) $\Rightarrow$ conj true

Q: Does A act properly on a CAT(0) cubical complex.

(4) FC-type
A_T finite type $\Leftrightarrow$ $s_i, s_j \in T \Rightarrow m_{ij} < \infty$ (no Euclidean or hyp. sub groups)

combinatorial normal forms ~ automatic structure
geometric $\mathcal{D}_A$ CAT(0) $\Rightarrow$ conj true
Q: Conj. $\mathcal{D}_A$ ~~exists~~ hyperbolic $\Leftrightarrow$ $\mathcal{D}_{W}$ hyperbolic

(5) RAAG's
A is right angled if $m_{ij} = 2, \infty$ $\forall i,j$