Surfaces in 3-manifolds

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The aim of this talk is to demonstrate that surfaces in 3-manifolds are very, very useful.

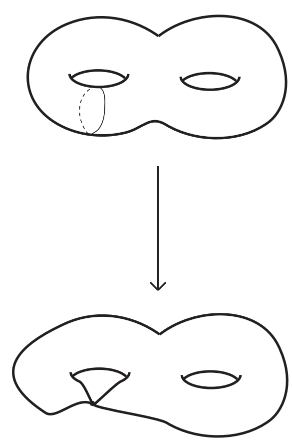
Kneser showed in 1930 that if M^3 is a compact, connected, orientable 3-manifold then

$$M^3 = P_1 \# P_2 \# ... \# P_n$$

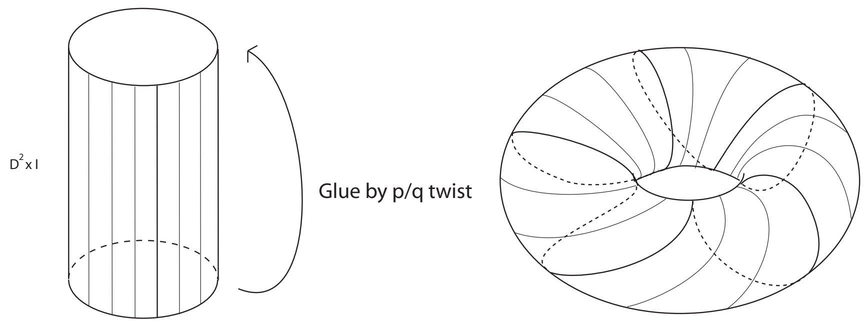
where each P_i is prime. This decomposition is unique up to order and S^3 summands.

A two sided surface $S \subset M^3$ is *incompressible* if every curve in S which bounds a disk in M^3 also bounds a disk in S (it is also required that it has no S^2 components and is not boundary parallel, but we won't worry about that here).

The following embedding of a genus 2 surface in $\mathbb{R}^3$ is compressible, as the curve drawn bounds a disk in $\mathbb{R}^3$ but not in the surface.



A *Seifert fibered space* is a 3-manifold that can be decomposed into a disjoint union of circles in such a way that each circle has a neighborhood that looks as follows:



M^3 is *geometrically atoroidal* if every incompressible T^2 is boundary parallel. Jaco-Shalen, and Johannson proved in 1979 that if M^3 is a compact orientable irreducible 3-manifold, there exists a collection T of tori in M^3 such that $M^3 \backslash T$ has components that are either atoroidal or Seifert fibered, and a minimal such collection is unique up to isotopy.

Note that a space can be simultaneously Seifert fibered and geometrically atoroidal, for example a product of a pair of pants and a circle.

Let χ be a simply connected Riemannian manifold. M^3 is said to have a geometric structure modeled on χ if there is a discrete group of isometries Γ such that $M^3 \cong \chi/\Gamma$.

Thurston conjectured that every closed 3-manifold M^3 admits a decomposition into pieces that are geometric. The decompositions required are of the two types mentioned above: prime decompositions, and torus decompositions (a variant of the JSJ torus decomposition). Work of Perelman has proved this conjecture.

The most important geometry for the study of 3-manifolds is hyperbolic geometry, as "most" pieces of a manifold are homeomorphic to $\mathbb{H}^3/\Gamma$, for $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$.

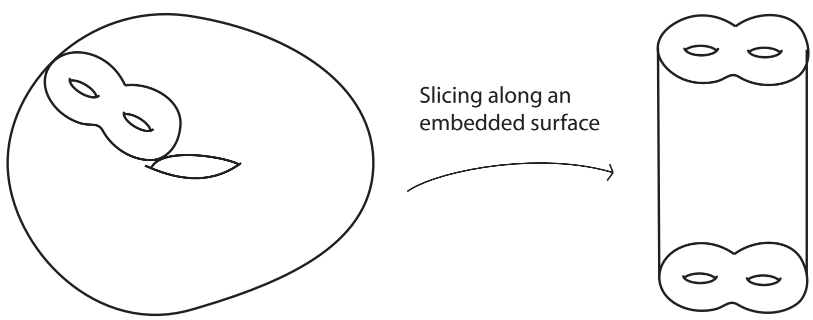
M^3 is said to be *Haken* if it is irreducible, orientable and contains a 2-sided incompressible surface. Thurston was able to prove the geometrization conjecture for Haken manifolds. Note that not all hyperbolic 3-manifolds are Haken. Haken showed that these manifolds satisfy a heirarchy, i.e. we can cut along incompressible surfaces and end up with a presentation of the manifold as a union of balls with disjoint interiors. Waldhausen showed that closed Haken manifolds with the same fundamental group are homeomorphic.

As a point of terminology, we say that a space is *virtually* "X" if it has a finite sheeted cover that is "X," and that a group is *virtually* "X" if it has a finite index subgroup that is "X". Waldhausen conjectured that any irreducible 3-manifold with infinite fundamental group is virtually Haken. It is known that virtually Haken manifolds are geometric, so this conjecture implies geometrization for irreducible 3-manifolds with infinite π_1. This conjecture is still open for hyperbolic manifolds. There is tons of evidence to support this conjecture, for example Dunfield and Thurston showed that all of the hyperbolic manifolds in the SnapPea census (around 10,000 manifolds) are virtually Haken.

If $H_2(M^3) \neq \varnothing$ (or by duality if $H^1(M^3) \neq 0$) then M^3 contains a closed incompressible surface. Stallings proved that given a group homomorphism $\phi : \pi_1(M) \to \mathbb{Z}$, there is a continuous map $p : M^3 \to S^1$ such that $p^* = \phi$ and that for a generic point x, $p^{-1}(x)$ is an incompressible surface. With this theorem in hand, a natural approach to showing that a manifold is virtually Haken is to look for finite index subgroups with infinite abelianization, as the we can easily define a map from the cover corresponding to this subgroup to $\mathbb{Z}$ and use Stallings' result.

Recall that the first Betti number b_1 of a topological space X is the rank of the abelian group $H_1(X)$. It is conjectured that for every irreducible 3-manifold M^3 with infinite fundamental group, for every K, there exists a finite sheeted cover $\widetilde{M^3}$ such that $b_1(\widetilde{M^3}) > K$. This is called the virtually infinite b_1 conjecture. This clearly implies the "virtually positive b_1 conjecture": that every such M^3 has a finite sheeted cover with positive first Betti number. This is known to imply the virtually Haken conjecture. Cooper, Lang, Reid and Agol have proven that the virtually positive b_1 conjecture and the virtually infinite b_1 conjecture are equivalent for closed, arithmetic, hyperbolic 3-manifolds.

A manifold is called fibered if it is homeomorphic to a surface bundle over the circle, i.e. if there is a map $p : M^3 \to S^1$ such that the preimage of every point in S^1 is homeomorphic to the same surface at every point.



It is conjectured that every hyperbolic 3-manifold is virtually fibered. In order to think about this conjecture it is important to understand non-separating surfaces in a manifold M^3. One way to do this is was introduced by Thurston, and is called the Thurston norm. Let S be a connected orientable surface, and let $x(S) := \max(-\chi(S), 0)$. If S_i is a connected orientable surface for $1 \leq i \leq n$ and $S = \cup_{i=1}^n S_i$, $x(S) := \sum_{i=1}^n x(S_i)$. We can use this to define a function on the cohomology of a compact manifold (possibly with boundary) by $x(\alpha) := \min\{x(S) \mid [S] = \alpha^*\}$ for $\alpha \in H^1(M^3)$ and α^* is the dual (either Poincaré or Lefschetz depending on whether $\partial M = \varnothing$) of α. We can extend this function to $H^1(M^3, \mathbb{Q})$ by linearity and to $H^1(M^3, \mathbb{R})$ by continuity. Thurston showed that this defines a seminorm on $H^1(M^3, \mathbb{R})$, which is in fact a norm if M^3 is hyperbolic.

This norm relates to fibrations in the following way. Thurston showed that the unit ball in this norm is a polyhedron with finitely many faces. The ray from a cohomology class α to zero passes through some face F_α of this polyhedron. If α is a lattice point that is represented by a fibration (i.e. if its dual can be represented by is an incompressible surface that is a fiber in some fibration of M^3) then Thurston showed that any other lattice point α' such that the ray from zero to α' passes through F_α is also represented by a fibration. Such a face is called a "fibered face."

One question relating to fibered faces is where the number goes up as we take finite sheeted covers. Long and Reid have shown that if M^3 is fibered, closed and arithmetic, then for any K there exists a finite sheeted cover of M whose unit norm ball has more than K fibered faces. Long and Reid have also formulated the following characterization of arithmeticity for finite volumer hyperbolic 3-manifolds: M^3 is arithmetic $\iff$ for all closed geodesics γ in M^3 there exists a finite sheeted cover $M_\gamma \to M$ such that M_γ admits an orientation preserving involution Σ, and the fixed point set of Σ contains a component of the preimage of γ.

Agol has shown that manifolds whose fundamental group satisfy condition RFRS (a complexity condition for π_1, satisfied, for example, by right angled Coxeter groups) are virtually fibered.

Many "virtual" questions can be rephrased in terms of commensurability. M_1 and M_2 are said to be commensurable if they share a finite sheeted cover. In the case of two hyperbolic manifolds $M_1 \cong \mathbb{H}^3/\Gamma_1$, $M_2 \cong \mathbb{H}^3/\Gamma_2$, this is equivalent to Γ_1 and some conjugate of Γ_2 sharing a finite index subgroup. Commensurability is an equivalence relation, and any property that is preserved under taking finite sheeted covers can be considered as a property of the commensurability class. As an example, being virtually fibered is a property that is shared by all members of a given commensurability class.

Thus a classification of hyperbolic 3-manifolds up to commensurability would answer many of the questions asked above. As an example of results in this direction, it is known that torus knot complements are all commensurable and that hyperbolic 2-bridge knot complements are pairwise incommensurable. In fact, hyperbolic 2-bridge knot complements are not commensurable with any other knot complement.