Automorphisms of RAAGs and Partially symmetric automorphisms of free groups

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joint work with R. Charney
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Right-angled Artin groups

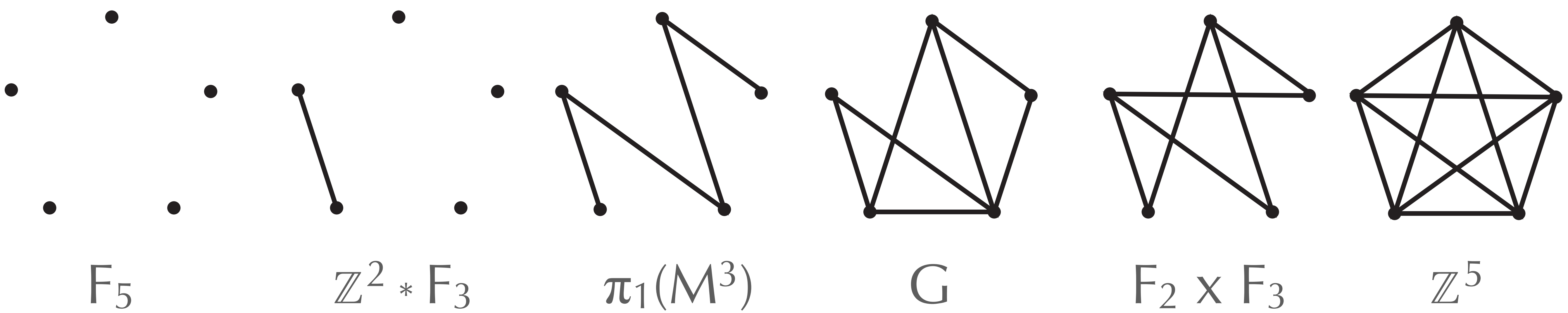
Γ = simplicial graph

The *right-angled Artin group* A_Γ is given by:

Generators: nodes of Γ

Relators: vw = wv if [v,w] is an edge of Γ

Examples:



Theorem [Droms] A_Γ is a 3-manifold group if and only if Γ is a disjoint union of trees and triangles.

Automorphisms of RAAGs

Basic question: To what extent does Out(A_Γ) share properties with Out(F_n) and GL($n, \mathbb{Z}$)?

e.g. Finiteness properties

Does Out(A_Γ) always have finite virtual cohomological dimension? If so, what is it?

Does it even have torsion-free subgroups of finite index?

Is it finitely presented? Does it have finitely generated homology?

Etc.

Automorphisms of RAAGs

Theorem [Laurence, Servatius] Aut(A_Γ) is generated by automorphisms of types (1)-(5):

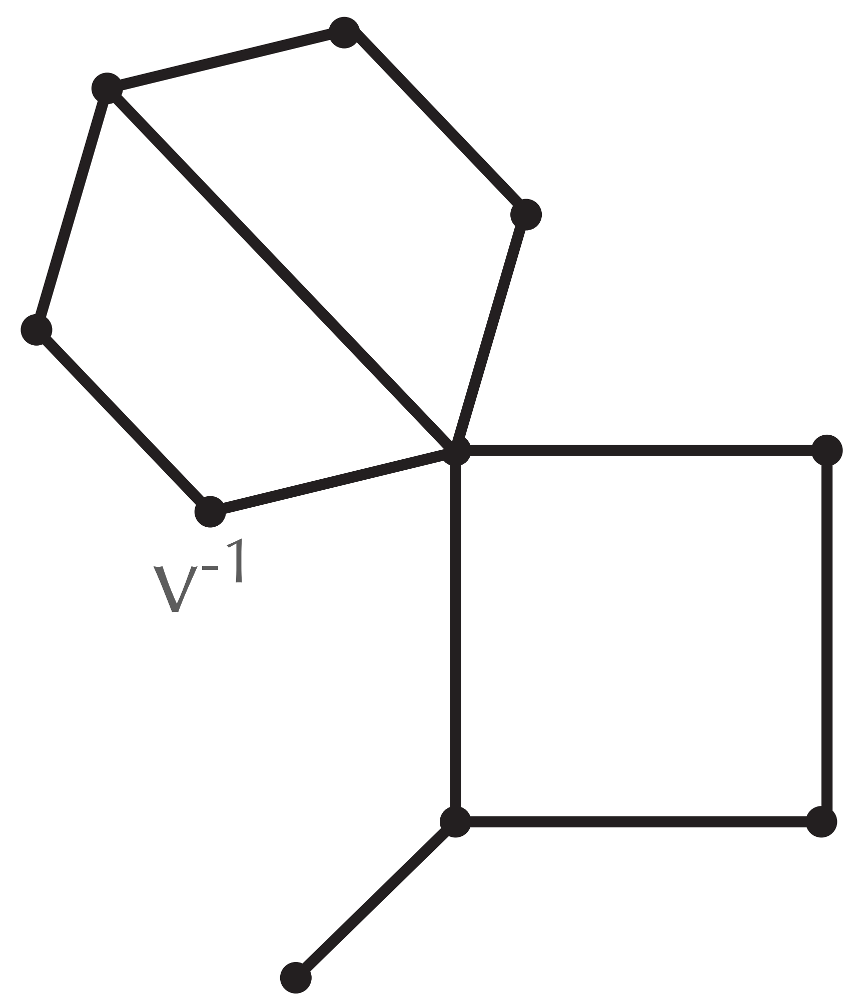
(1) (Inner automorphisms)

(2) Inversions in nodes

(3) Graph automorphisms

(4) Partial conjugations

(5) Transvections



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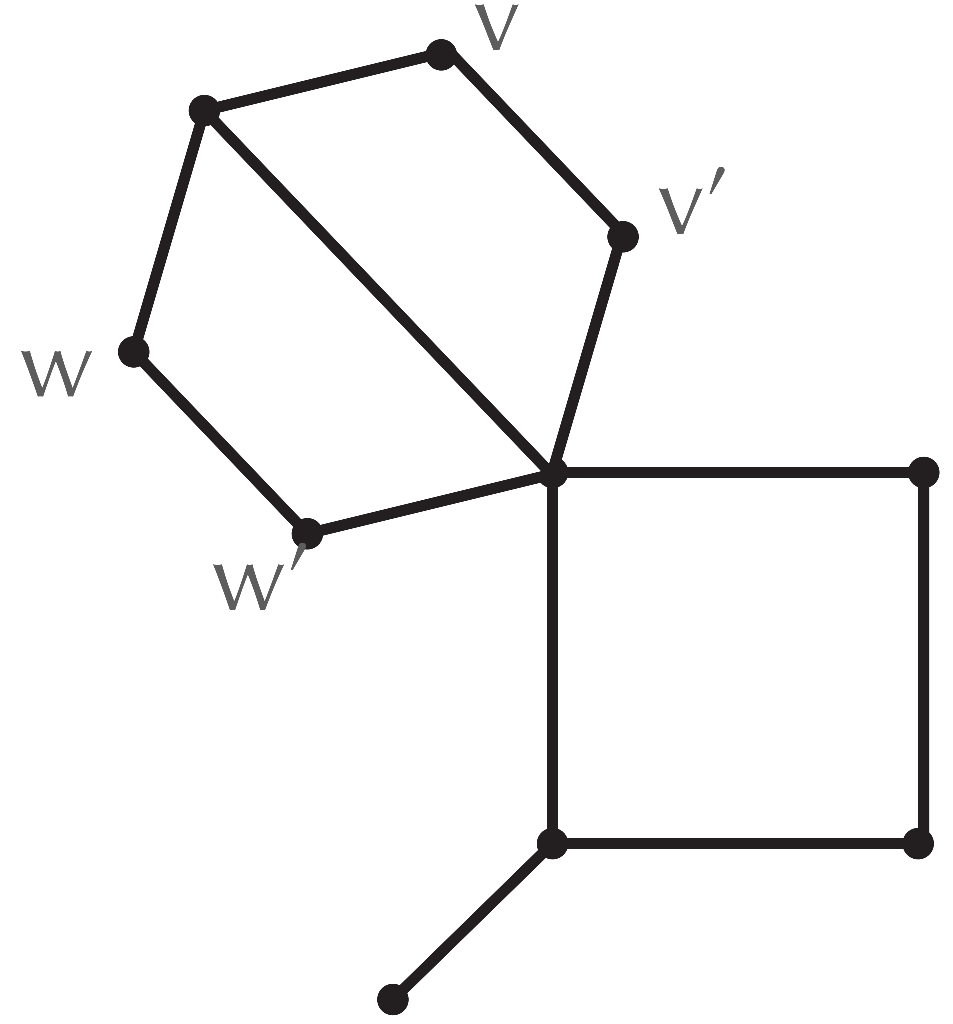

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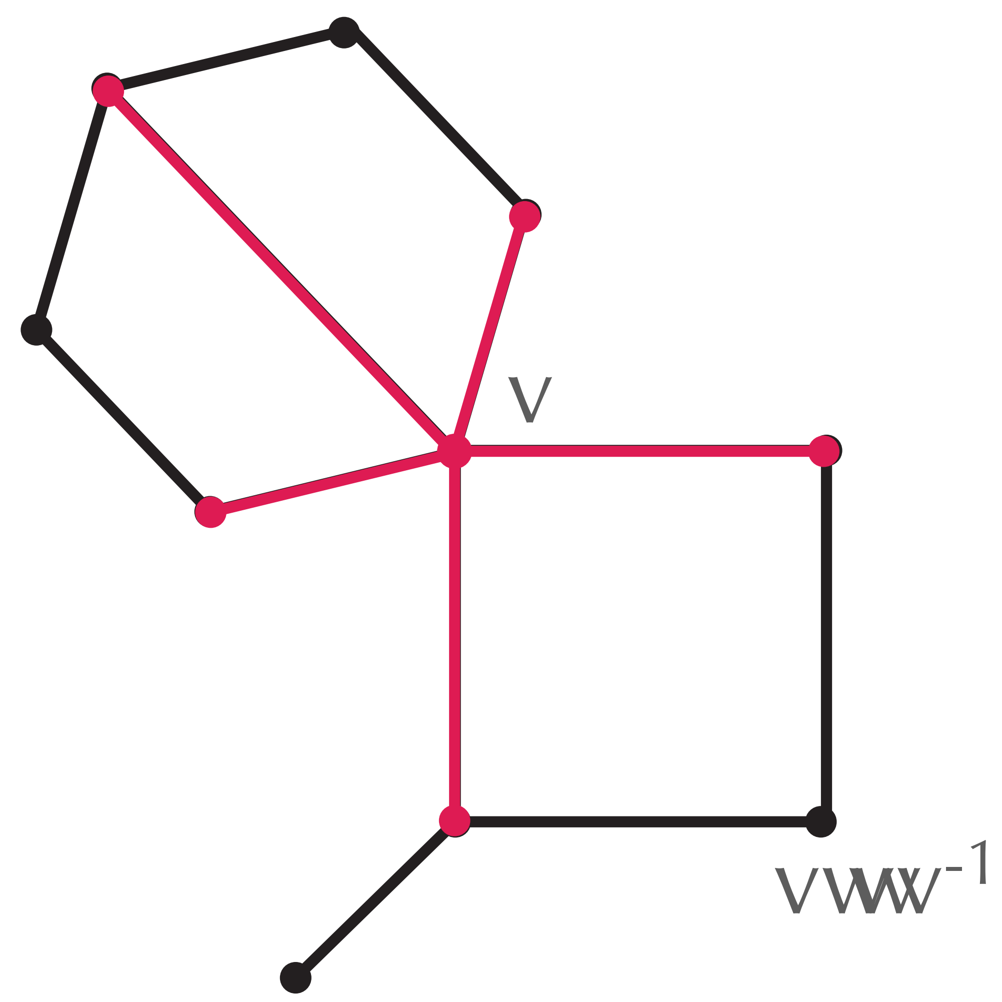
If st(v) separates Γ, conjugate one component by v



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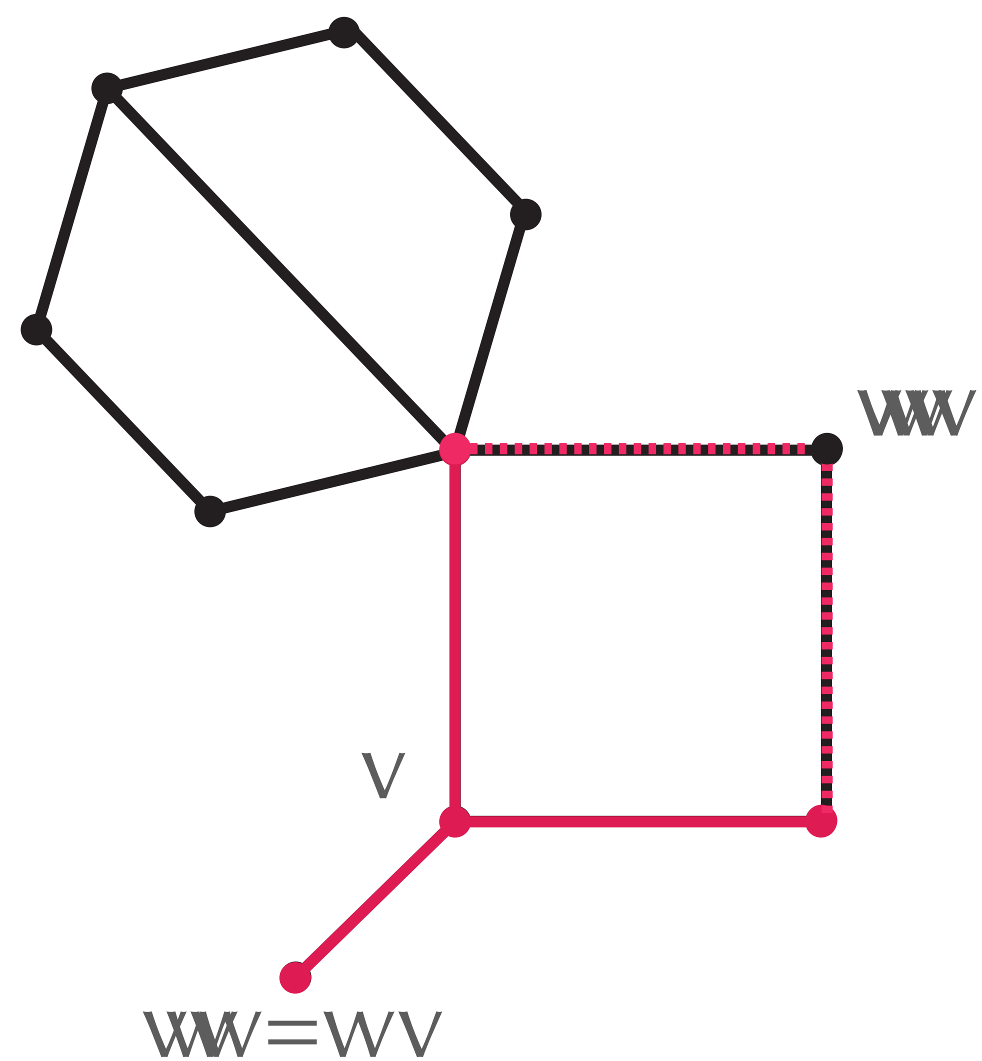
If lk(w) $\subset$ st(v)   map w $\mapsto$ vw

(or, map w $\mapsto$ wv)

If [v,w]=1, these are the same.



Automorphisms of RAAGs

Theorem [Laurence, Servatius] $\mathrm{Aut}(A_\Gamma)$ is generated by automorphisms of types (1)-(5):

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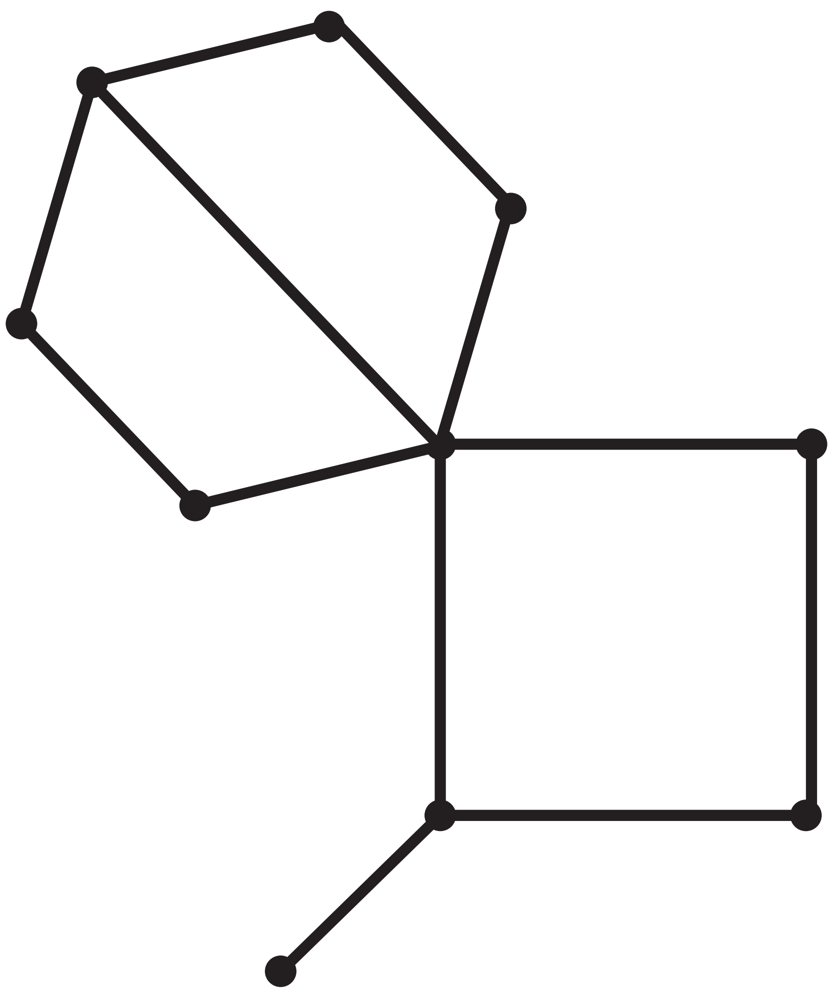

Definition. $\mathrm{Aut}^0(A_\Gamma) < \mathrm{Aut}(A_\Gamma)$ is the subgroup generated by inversions, transvections and partial conjugations.

Lemma. $\mathrm{Aut}^0(A_\Gamma)$ has finite index in $\mathrm{Aut}(A_\Gamma)$

Projection homomorphisms

Focus on $\mathrm{Out}(A_\Gamma)$, $\mathrm{Out}^0(A_\Gamma)$.

Define $w \le v$ if $\mathrm{lk}(w) \subset \mathrm{st}(v)$

$[v] = \{w \mid w \le v \text{ and } v \le w\}$ $L_v = \mathrm{lk}(v) - [v]$

Key tool: If v is maximal, there is a projection homomorphism $p_v\colon \mathrm{Out}^0(A_\Gamma) \to \mathrm{Out}(A_{L_v})$.

Case: Γ a tree.

v maximal $\Leftrightarrow$ v is is not a leaf
Then A_{L_v} is free, and the image of p_v is the subgroup sending non-leaf vertices to conjugates of themselves: $\mathrm{Im}(p_v) = P\Sigma(n,r) \subset \mathrm{Out}(A_{L_v})$

Projection homomorphisms

Theorem [Charney-V]: The kernel of the product

$$P = \prod p_v : \mathrm{Out}^0(A_\Gamma) \to \prod \mathrm{Out}(A_{L_v})$$

over all maximal vertices is free abelian of finite rank.

Case: Γ a tree. If Γ has *e* edges, rank(ker)= *e*-1.

Applications:

Theorem A: If Γ has no triangles, then Out(A_Γ) satisfies the Tits' alternative: every subgroup is either virtually solvable or contains a free nonabelian subgroup.

Projection homomorphisms

Theorem [Charney-V]: The kernel of the product

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Applications:

Theorem B: Out(A_Γ) is virtually torsion-free.

Theorem C: Out(A_Γ) has finite VCD.

Proof of Theorems B & C: Induction on rank(A_Γ), together with results of Guirardel-Levitt.

Virtual cohomological dimension

Lower bound:

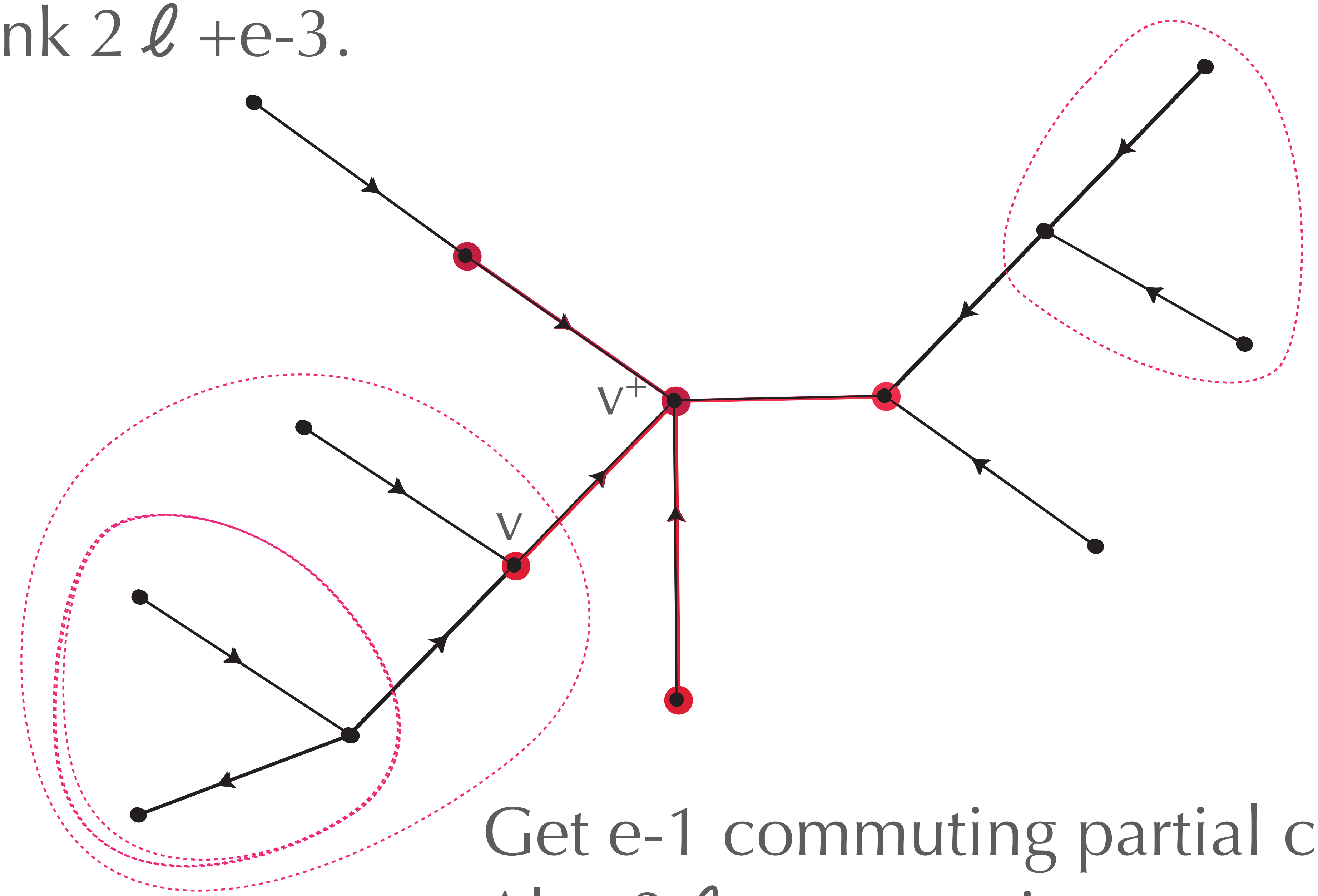
Out(A_Γ) contains free abelian subgroups generated by (1) non-interacting transvections and
(2) partial conjugations on disjoint subsets.

In general, don't expect this to equal the VCD (e.g. GL(n,$\mathbb{Z}$) has VCD ~ $n^2/2$ but abelian subgroups of rank at most ~ $n^2/4$.

If Γ has no triangles or squares, the rank of this subgroup is at least very close to the VCD.

Free abelian subgroups of Out(A_Γ)

Claim. If Γ is a tree with ℓ leaves and e edges, Out(A_Γ) contains a free abelian subgroup of rank 2ℓ+e-3.



Partially conjugate by v^+

Get e-1 commuting partial conjugations.
Also 2ℓ transvections onto leaves.

But we've included 2 inner autos

Virtual cohomological dimension

Upper bound:

Recall $\mathsf{P} = \prod \mathsf{p}_v : \mathsf{Out}^0(A_\Gamma) \to \prod \mathsf{Out}(A_{\mathsf{L}_v})$

where the product is over maximal v.

By standard theory, this gives

$\text{vcd}(\text{Out}(A_\Gamma)) \leq \text{rank}(\text{Ker}(P)) + \sum \text{vcd}(\text{Im}(p_v))$

Case: Γ a tree. We know

* rank(Ker(P)) = e-1
* v is maximal iff v is not a leaf
* $\text{Im}(p_v) \subset P\Sigma(n,r)$, where

  n = valence of v, r = no. of non-leaves at v

So we would like to compute $\text{vcd}(P\Sigma(n,r))$

Partially symmetric automorphisms

F_n free on generators $x_1,\ldots,x_n$

$P\Sigma(n,r)$ is the subgroup of $Out(F_n)$ consisting of automorphisms which send each x_i to a conjugate of itself, for $i \leq r$.

If $r > 0$, the automorphisms

$$x_i \mapsto x_i\, x_1 \text{ and } x_i \mapsto x_1\, x_i \quad (i>r) \text{ and}$$

$$x_i \mapsto x_1^{-1}\, x_i\, x_1 \; (2 \leq i \leq r)$$

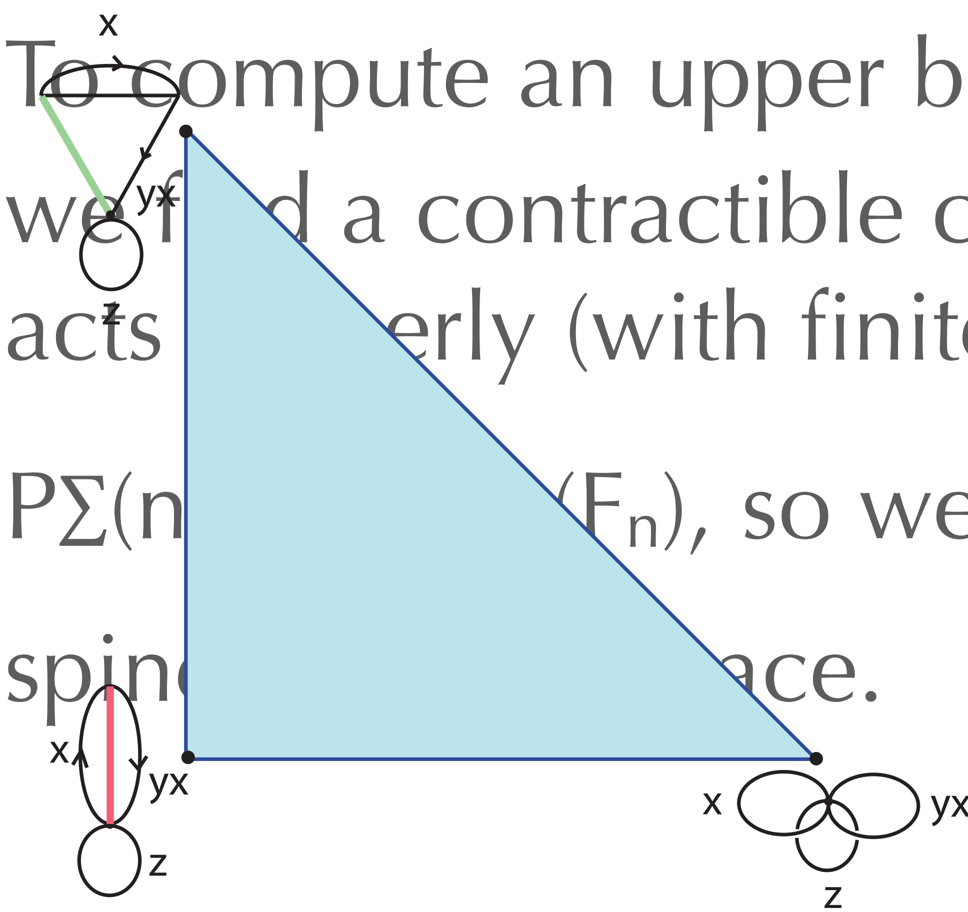
generate a free abelian subgroup of $P\Sigma(n,r)$ of rank $2n\text{-}r\text{-}2$.

Theorem[Bux-Charney-V]: The VCD of $P\Sigma(n,r)$ is equal to $2n\text{-}r\text{-}2$.

A complex for PΣ(n,r)

To compute an upper bound for the vcd of PΣ(n,r), we f d a contractible complex L(n,r) on which it acts erly (with finite stabilizers).

PΣ(n (F$_n$), so we *have* such a space: the spin ace.

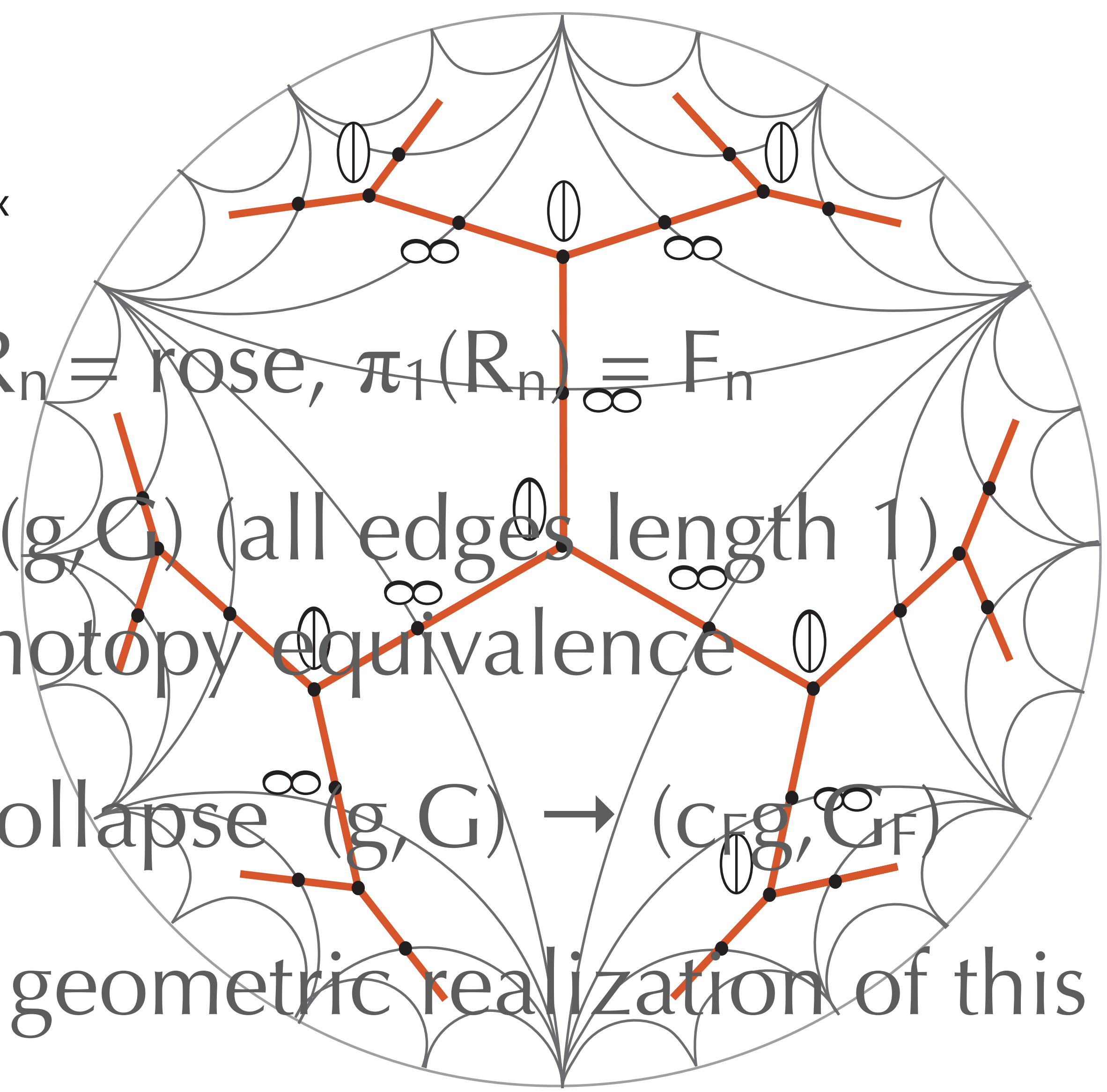


Spine of Outer space: R_n = rose, $\pi_1(R_n) = F_n$

Vertex = marked graph (g,G) (all edges length 1)
g:$R_n \to G$ a homotopy equivalence

Poset relation = forest collapse (g,G) → (c_Fg,G_F)

Spine of Outer space is geometric realization of this poset



A complex for $P\Sigma(n,r)$

Our free abelian subgroup had rank 2n-r-2. The dimension of the spine is 2n-3. Too big!

W = finite set of cyclic words in F_n

$\rho = (g,G)$ a rose, $g:R_n \to G$

For $w \in W$, measure $|w|_\rho$ = length of $g(w)$ (tightened)

Define $\|\rho\| = \sum_{w\in W} |w|_\rho$ $\qquad K_W = \bigcup_{\|\rho\| \text{ minimal}} st(\rho)$

Theorem [Culler-V] For any set W of cyclic words, Outer space deformation retracts to K_W.

Now notice: stab(W) acts on K_W

A complex for PΣ(n,r)

Set $W = \{x_1, \ldots, x_r\}$

Define $\|\rho\| = \sum_{w \in W} |w|_\rho$ $\qquad K_W = \bigcup_{\|\rho\| \text{ minimal}} st(\rho)$

stab(W)=Out(n,r) contains PΣ(n,r) with finite index,

so suffices to compute vcd(stab(W))

Theorem [Culler-V] For any set W of cyclic words, Outer space deformation retracts to K_W.

But: $\dim(K_W)=2n-3$. Still too big!

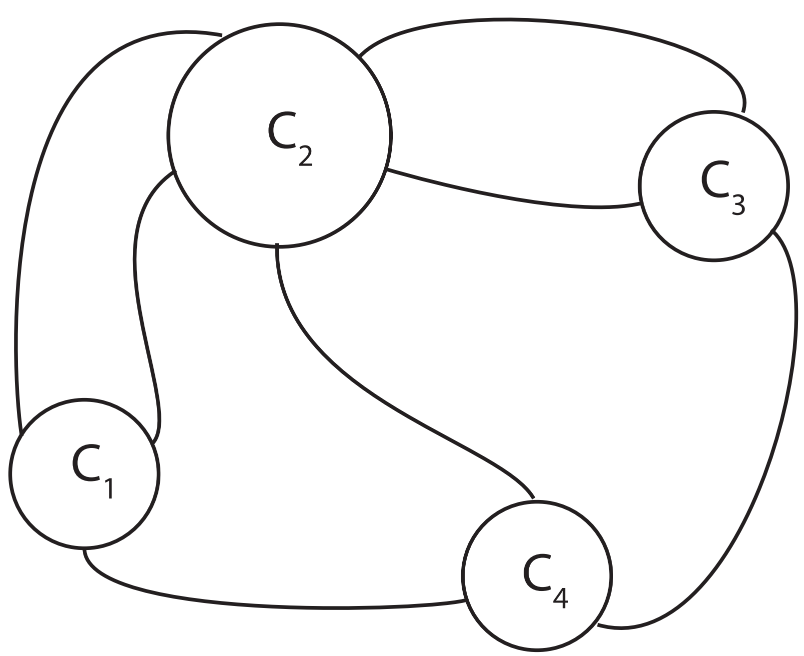
Now notice: stab(W) acts on K_W

A complex for PΣ(n,r)

We will retract K_W equivariantly onto a smaller subcomplex.

Lemma: $\gamma = (g,G)$ in K_W. Then $g(x_i)$ is a simple loop C_i, for $i \leq r$.

Definition: D(n,r) is the subcomplex of K_W spanned by (g,G) with disjoint C_i.



Exercise: dim(D(n,r)) = 2n-r-2

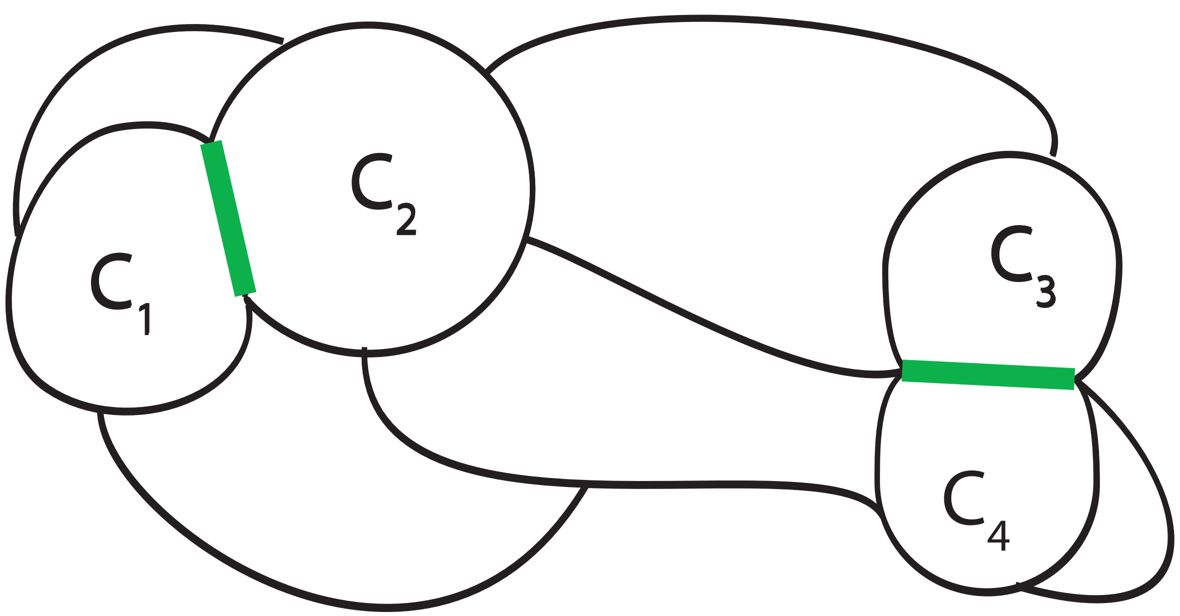
(hint: Euler characteristic)

A complex for PΣ(n,r)

We will retract K_W equivariantly onto a smaller subcomplex.

Lemma: $\gamma = (g,G)$ in K_W. Then $g(x_i)$ is a simple loop C_i, for $i \leq r$.

Let $F_\gamma = \cup\, C_i \cap C_j$.

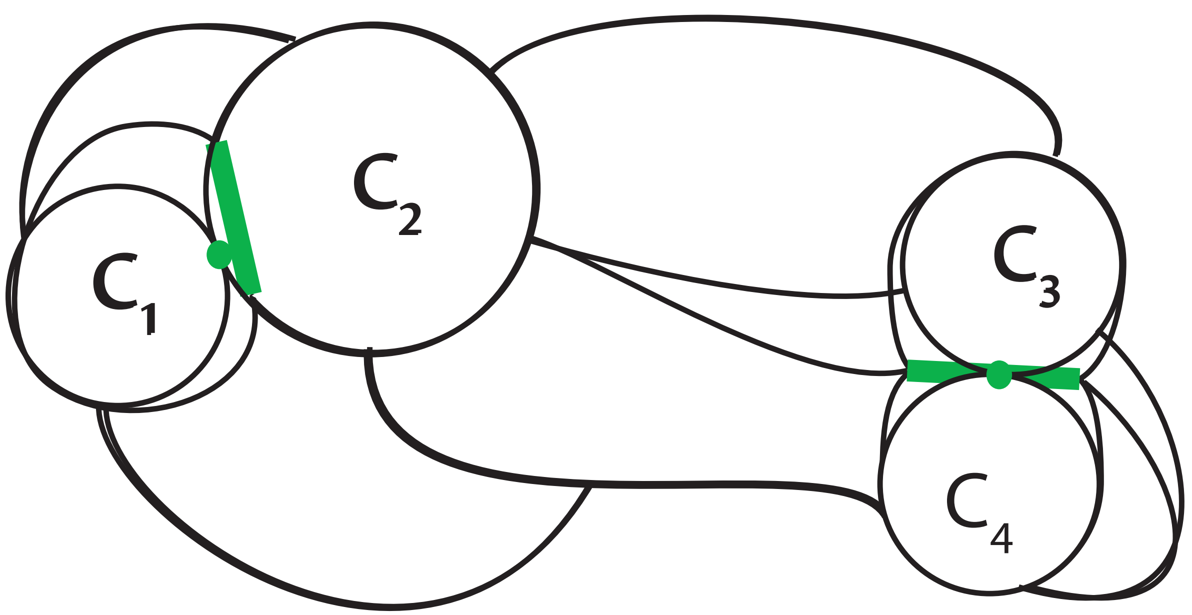


This is the *canonical forest.*

A complex for PΣ(n,r)

Proposition: Collapsing the canonical forest of every (g,G) gives a deformation retraction of K_W.

Let L(n,r) denote the image of this deformation retraction. In L(n,r), cycles C_i and C_j intersect at most in a point.



A complex for PΣ(n,r)

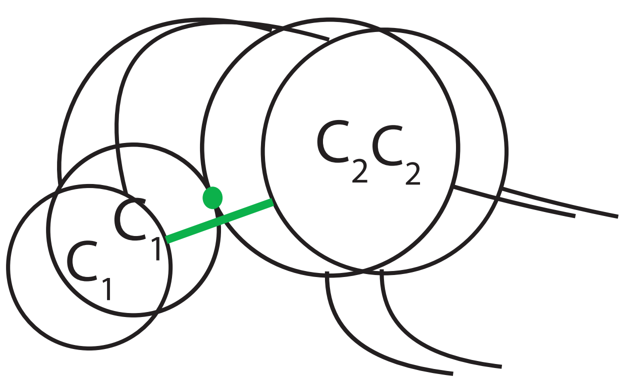
For r=n, get dim(L(n,n)) = n-2.

Corollary [Collins]: The VCD of PΣ(n,n) is equal to n-2.

If r<n, dim(L(n,r)) = 2n-3.

Still need to pull the cycles C_i apart.

If $C_i \cap C_j$ is an isolated node, only one way to do it:



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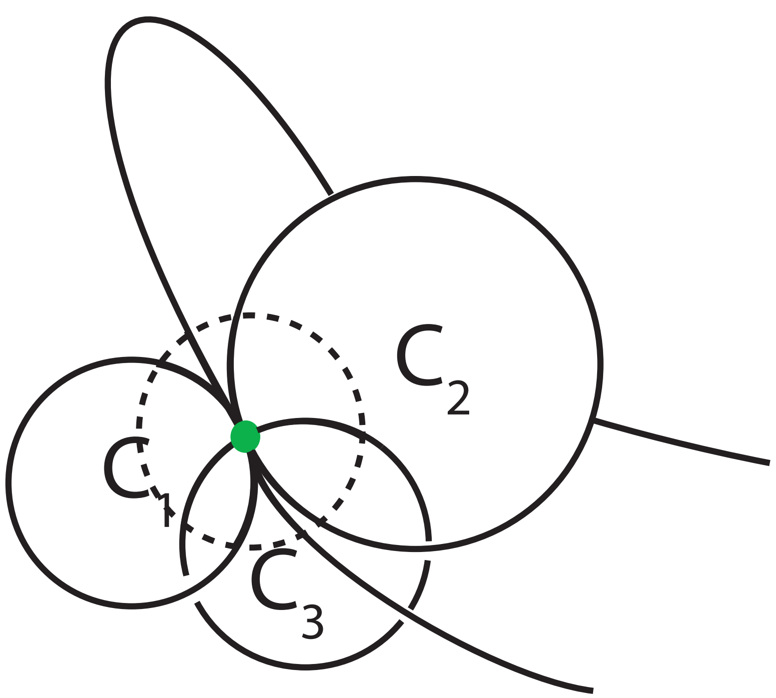
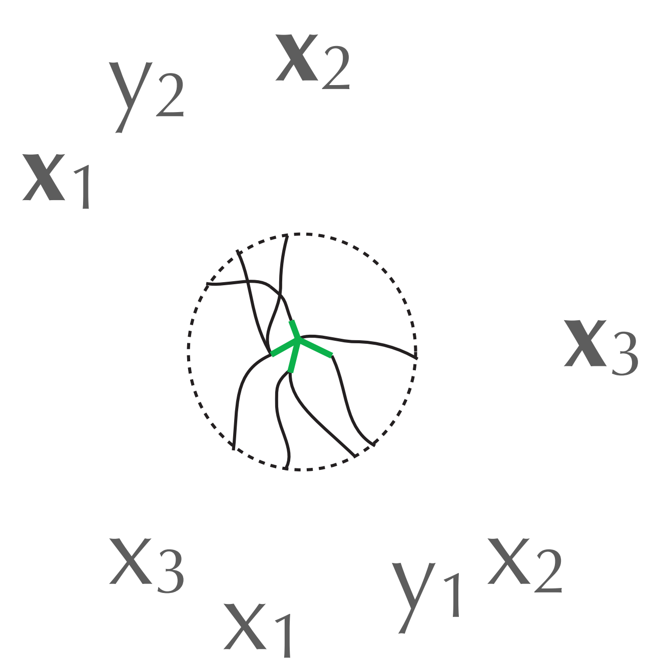
In general, there are many ways to do it:



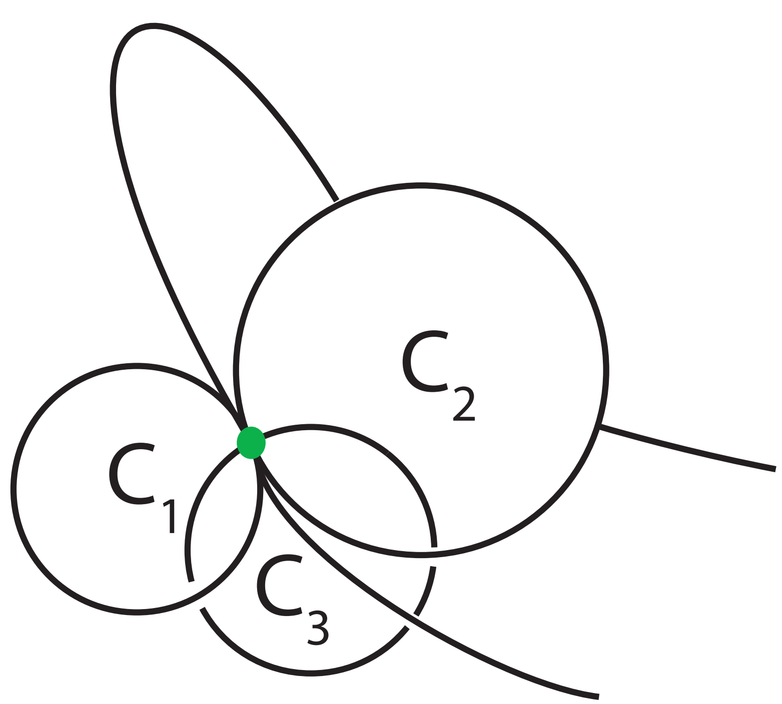


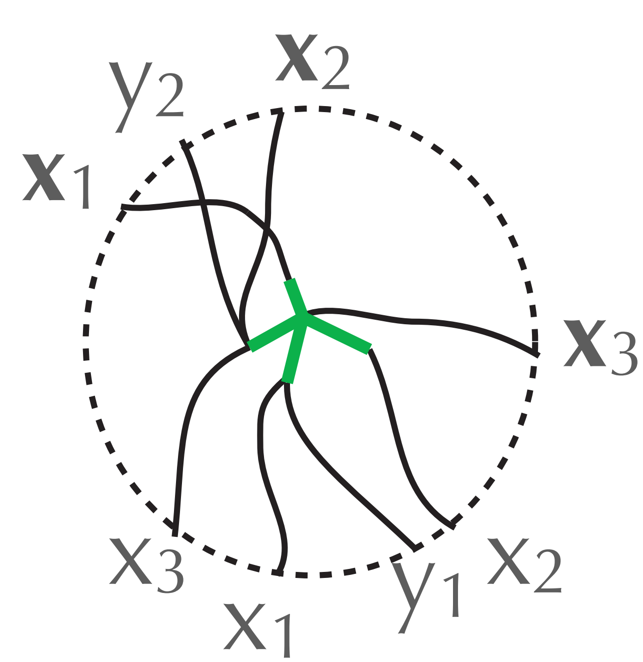
A complex for $P\Sigma(n,r)$

We can blow up the vertex into any tree, as long as each edge separates at most one x_i from $\mathbf{x}_i$

We prove the complex of all such blowups is contractible.

Combinatorial Morse theory then shows that $L(n,r)$ deformation retracts to $D(n,r)$.

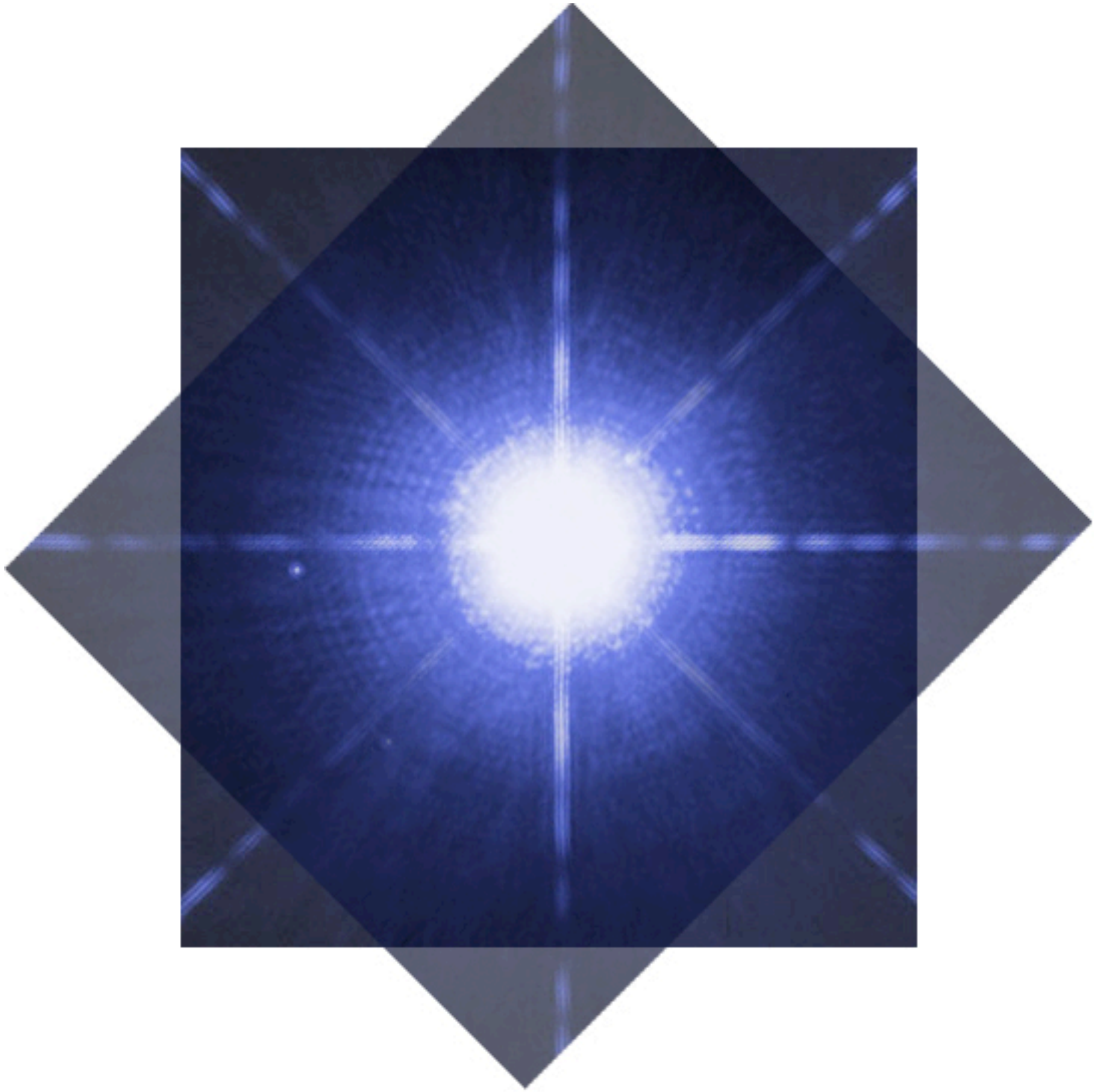




Contractibility of blowup space

Want to show the space of (legal) blowups of a stemmed rose is contractible

Blowing up a star



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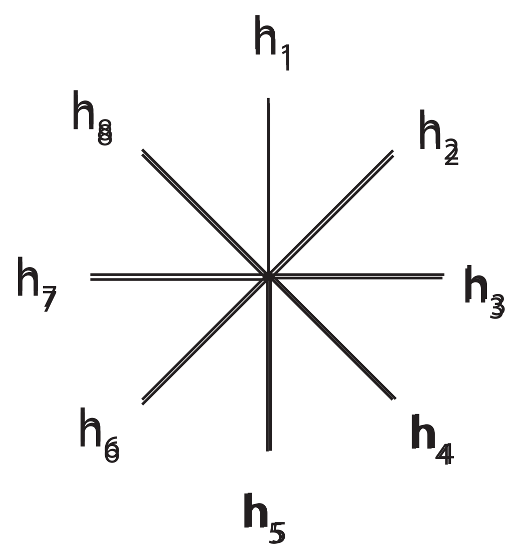
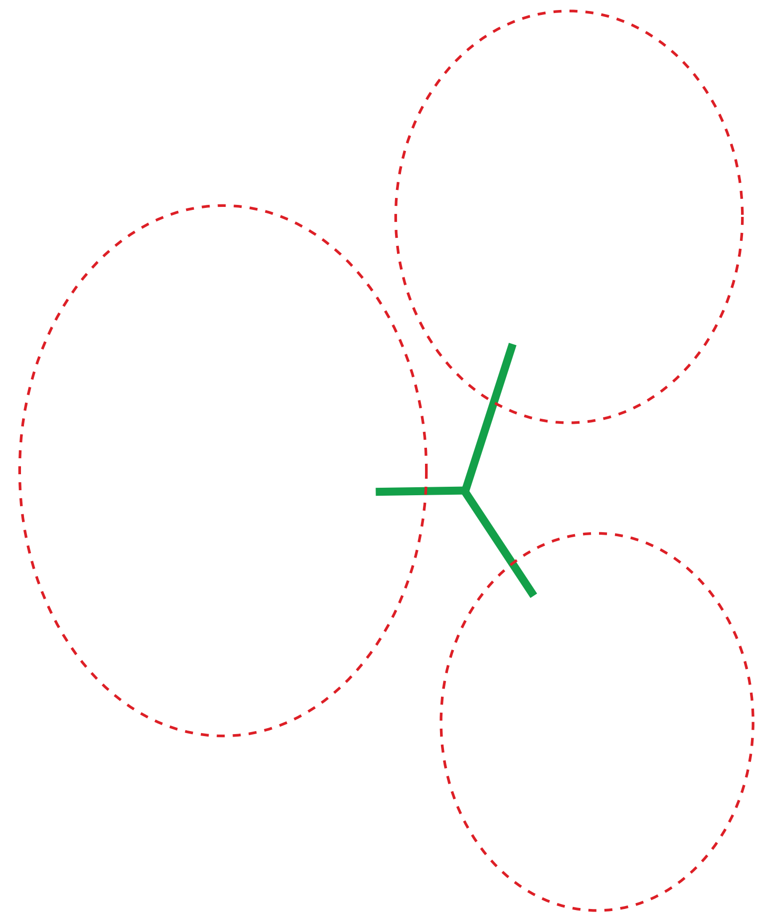




$H = \{h_1, \ldots, h_r\}$

Edges in **T** partition H

T = compatible set of partitions

$\mathcal{T}_r$ = simplicial complex

vertex = partition of H into two sets of cardinality ≥2

k-simplex = k+1 compatible partitions

Blowing up a star

Proposition: $\mathcal{T}_r$ is (r-3)-connected.

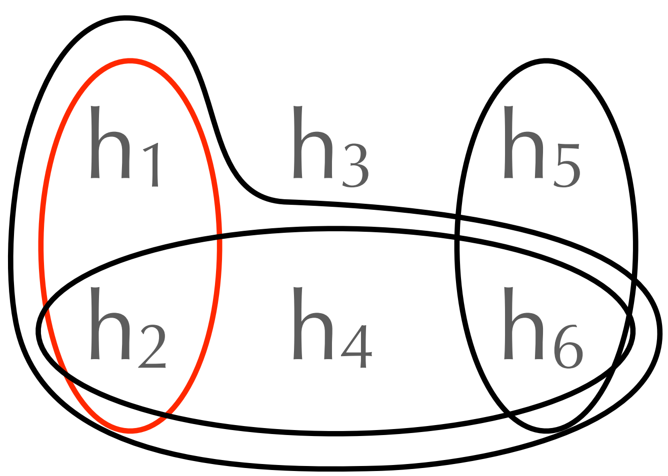
Proof: Construct a Morse function f

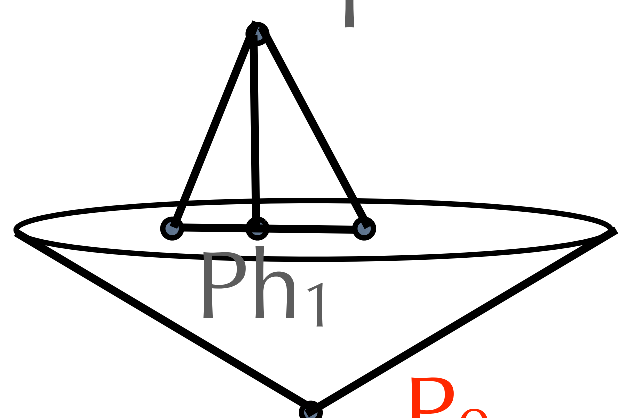
P_0 = $h_1, h_2 \mid h_3,\ldots,h_r$ ⇒ $f(P_0)=0$

P compatible with P_0 ⇒ $f(P)=1$

P separating h_1 from h_2 ⇒ $f(P)=\#(\text{side containing } h_2)$

Ph_1 : switch side of h_1





Exercise: If $1 < f(P) < r-2$, the descending link of P is a cone on Ph_1

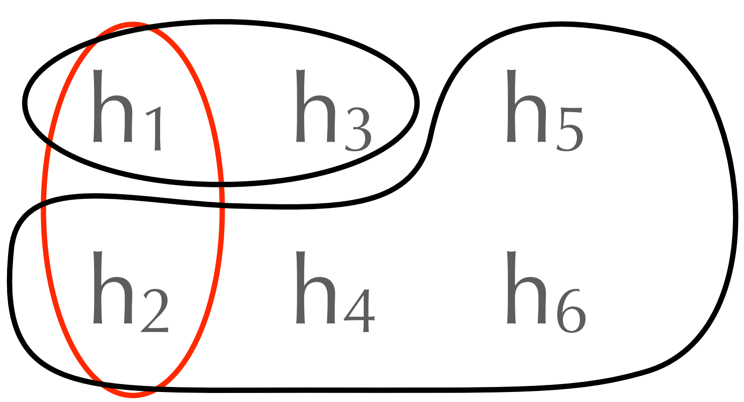
Blowing up a star

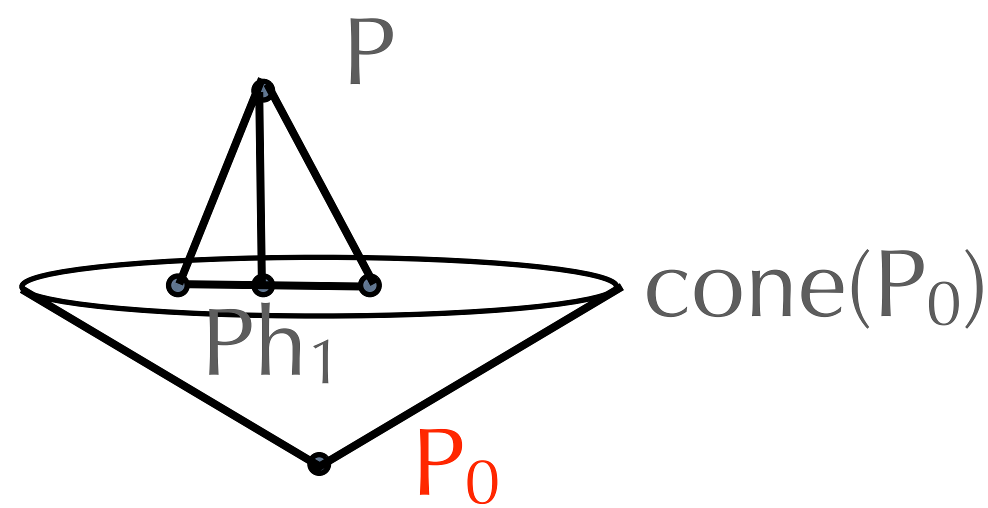
Proposition: $\mathcal{T}_r$ is (r-3)-connected.

Proof: Construct a Morse function

If f(P)= r-2

Claim: Descending link(P) = link(P) $\cong \mathcal{T}_{r-1}$





$\mathcal{T}_{r-1}$ is (r-4)-connected by induction. QED

VCD of Out(A_Γ) for Γ a tree

We now have

$$\mathrm{vcd}(\mathrm{Out}(A_\Gamma)) \leq \mathrm{rank}(\mathrm{Ker}(P)) + \sum \mathrm{vcd}(\mathrm{Im}(p_v))$$

$$\leq (e-1) + \sum_{v \text{ interior}} \mathrm{vcd}(P\Sigma(|v|,|v|_{int}))$$

$$= (e-1) + \sum_{v \text{ interior}} 2|v|-(|v|_{int})-2$$

$$= 2\ell + e - 3$$

(using the Euler characteristic of a tree)

This is agrees with our lower bound!

Reprise

To study automorphism groups of RAAGs we

(1) Proved that they have tffi subgroups

(2) Established upper and lower bounds on their VCD

For a tree-based RAAG we

(1) Found a free abelian subgroup of large rank

(2) Established an upper bound based on the VCD of $P\Sigma(n,r)$

(3) Since these bounds coincide, it sufficed to establish the VCD of $P\Sigma(n,r)$

Reprise

To study $P\Sigma(n,r)$ we

(1) Enlarged it to $Out(n,r) = stab_{Out(Fn)}\{x_1,\ldots,x_r\}$

(2) Used [CV] to find a contractible subcomplex K_W of the spine of outer space with Out(n,r) action.

(3) Retracted K_W to a subspace P(n,r)

(4) Retracted P(n,r) to D(n,r) using Morse theory

The dimension of D(n,r) is 2n-r-2, giving

$$VCD(P\Sigma(n,r)) = 2n\text{-}r\text{-}2$$

Reprise

We've established

Theorem: The VCD of the group of outer automorphisms of a RAAG based on a tree with e edges and ℓ leaves is equal to $2e+\ell-3$.

These methods also give upper and lower bounds for the VCD of $Out(A_\Gamma)$ in general, which sometimes agree.