Note Taker Checklist Form -*MSRI*

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Talk Title and Workshop assigned to:
Lattices acting on polyhedral complexes

Lecturer (Full name): ~~Ann~~ Anne Thomas
Date & Time of Event: November 6, 2007 11:30 AM - 12:20 PM

Check List:

( ) Introduce yourself to the lecturer prior to lecture. Tell them that you will be the note taker, and that you will need to make copies of their own notes, if any.

( ) Obtain all presentation materials from lecturer (i.e. Power Point files, etc). This can be done either before the lecture is to begin or after the lecture; please make arrangements with the lecturer as to when you can do this.

( ) Take down all notes from media provided (blackboard, overhead, etc.)

( ) Gather all other lecture materials (i.e. Handouts, etc.)

( ) Scan all materials on PDF scanner in 2nd floor lab (assistance can be provided by Computing Staff) – Scan this sheet first, then materials. In the subject heading, enter the name of the speaker and date of their talk.

Please do **NOT** use **pencil** or colored pens other than black when taking notes as the scanner has a difficult time scanning pencil and other colors.

Please fill in the following after the lecture is done:

1. List 6-12 lecture keywords:____________________

2. Please summarize the lecture in 5 or less sentences.

Once the materials on check list above are gathered, please scan ALL materials and send to the Computing Department. Return this form to Larry Patague, Head of Computing (rm 214)

Lattices acting on polyhedral complexes

Angela Barnhill
Northwestern University

Anne Thomas
Cornell University

OUTLINE

1. Definitions and questions
2. Classical results
3. Lattices in automorphism groups of polyhedral complexes
   a) Davis–Moussong complexes
   b) right-angled buildings
4. Open questions

Basic definitions

G locally compact top. gp

μ Haar measure

A subgroup $\Gamma \leq G$ is a <u>lattice</u>
if

- Γ is discrete
- $\mu(\Gamma \backslash G) < \infty$

Γ is <u>uniform</u> if $\Gamma \backslash G$ compact.

Questions Given G, determine

1. Existence

of unif./nonunif. lattices in G.
How to construct?

2. Covolumes

$$\mathcal{V}(G) = \{\mu(\Gamma \backslash G) \mid \Gamma \text{ lattice in } G\}$$

Is $\mathcal{V}(G)$ discrete?

3. Commensurators (with A. Barnhill)

of unif. lattices in G.

$$\mathrm{Comm}_G(\Gamma) = \{g \in G \mid \Gamma \cap g\Gamma g^{-1} \text{ finite index in } \Gamma \text{ and } g\Gamma g^{-1}\}$$

Is $\mathrm{Comm}_G(\Gamma)$ dense in G?

Classical results

Existence

G noncompact simple real Lie gp   e.g. $PSL_n(\mathbb{R})$

Borel: G admits unif. and nonunif. lattices

Margulis: in higher rank, all lattices arithmetic

Covolumes

H alg. gp over nonarch. local field   e.g. $PSL_n(\mathbb{Q}_p)$

Borel: in higher rank, $\forall c > 0$ $\exists$ only finitely many lattices $\Gamma \leq H$ with $\mu(\Gamma \backslash H) < c$.

$\Rightarrow \mathcal{V}(H)$ discrete.

Commensurators

Margulis: Γ arithmetic $\Leftrightarrow$ $\mathrm{Comm}_G(\Gamma)$ dense in G.

Automorphism groups of polyhedral complexes

X = locally finite polyhedral complex

$G = \mathrm{Aut}(X)$ locally compact gp

When is G nondiscrete?

If $\exists$ $g_n \in G$, $g_n \neq 1$ fixing $\mathrm{Ball}(n) \subseteq X$ then G is nondiscrete.

e.g. X = locally finite tree

Motivation:

Theorem (Tits) X = building for $\mathcal{G}$ higher rank alg. gp over nonarch. local field e.g. $\mathcal{G} = PSL_3(\mathbb{Q}_p)$. Then $\mathcal{G}$ finite index or cocompact in $\mathrm{Aut}(X)$.

Combinatorial volume formula

Theorem (Serre)
Let $G = \mathrm{Aut}(X)$, X locally finite poly. cx.
If $G \backslash X$ compact then Haar measure μ on G may be normalised s.t. for all discrete $\Gamma \leq G$

$$\mu(\Gamma \backslash G) = \sum_{x \in V(\Gamma \backslash X)} \frac{1}{|\Gamma_x|}$$

Hence:
Γ a <u>lattice</u> $\Longleftrightarrow$ series converges.

Moreover:
Γ a <u>unif. lattice</u> $\Longleftrightarrow$ $\Gamma \backslash X$ compact.

go to board

Existence and covolumes of lattices for Davis-Moussong complexes

Theorem (T)

Let $X = X_W$ be the Davis-Moussong complex for a Coxeter group W.

Let $G = Aut(X)$.

Suppose

- G is nondiscrete (Haglund-Paulin)
- 2 more technical conditions hold (If $\dim(X)=2$, enough that all $m_{ij}=m$ even, and that $Aut(L)$ transitive on vertices of L.) where L = finite nerve of W

Then G admits

1. a nonuniform lattice Γ
2. an infinite family of unif. lattices (Γ_n) s.t. $\mu(\Gamma_n \backslash G) \longrightarrow \mu(\Gamma \backslash G)$.

Hence $\mathcal{V}(G)$ is nondiscrete.

Density of commensurators for right-angled buildings

Theorem (Barnhill-T, Haglund)
Let X be a right-angled building s.t. $G = Aut(X)$ nondiscrete.
Let $\Gamma_0 \leq G$ be the "standard unif. lattice".
Then $Comm_G(\Gamma_0)$ is dense in G.

Corollary
If $\dim(X) = 2$, $Comm_G(\Gamma)$ dense in G for all unif. lattices $\Gamma \leq G$.

Proof Γ commens. to Γ_0 by thm of Haglund.

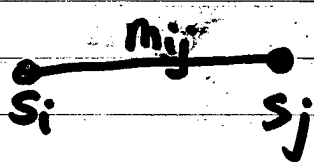
Coxeter groups + Davis-Moussong cxs

$W = \langle s_i, i \in I \mid (s_i s_j)^{m_{ij}} = 1 \rangle$

$m_{ii} = 2$, $m_{ij} \geq 2$ for $i \neq j$

Let L be <u>finite nerve</u> of W i.e. simplicial cx with

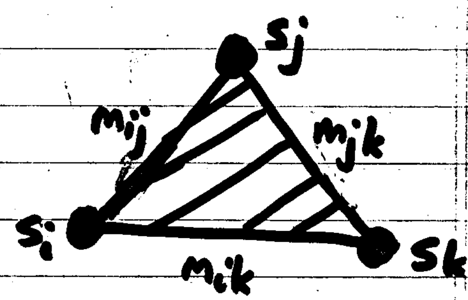
* vertices $\bullet\, s_i$
* edges



$m_{ij} < \infty$

$\iff W_{\{i,j\}} := \langle s_i, s_j \rangle$ is finite

* 2-simplices



$\iff W_J = \langle s_i, s_j, s_k \rangle$ finite for $J = \{i, j, k\} \subseteq I$

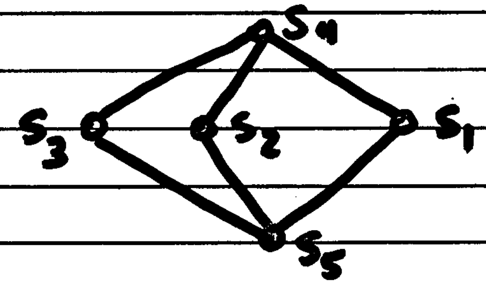
* etc

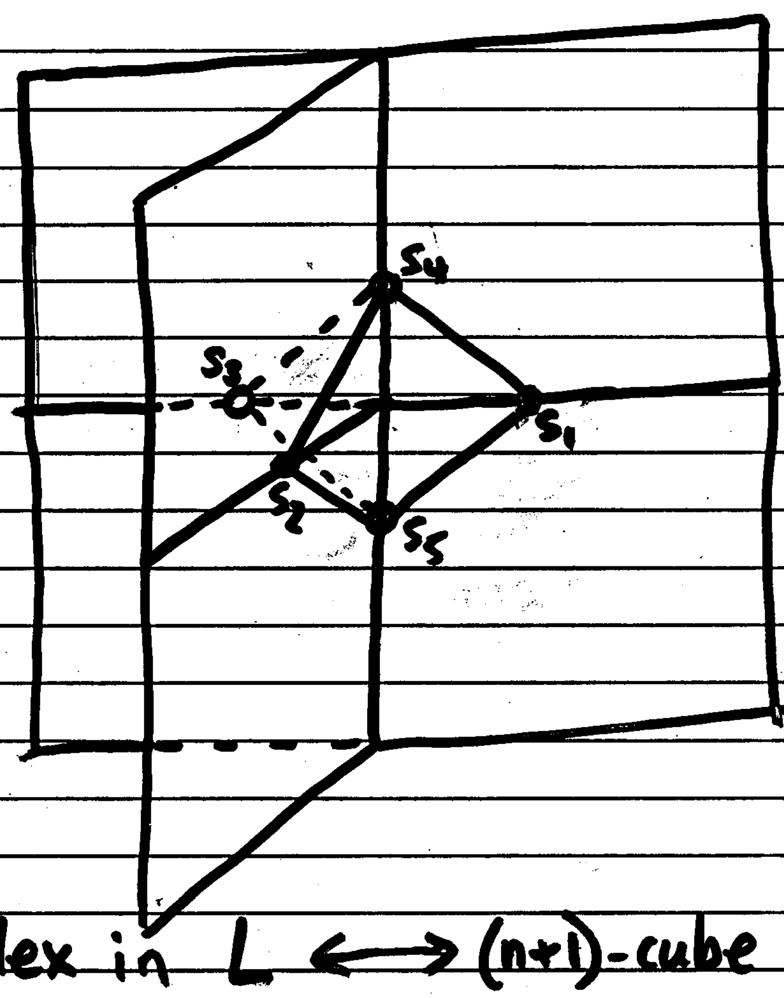
<u>Spherical subgroups</u> of W are $W_J = \langle s_j, j \in J \subseteq I \rangle$ which are <u>finite</u>.

Davis-Moussong complexes

W Coxeter gp, L = finite nerve of W

Let $C(L)$ be the cubical cone on L
(Davis-Januszkiewicz-Scott).

e.g. $L =$ 

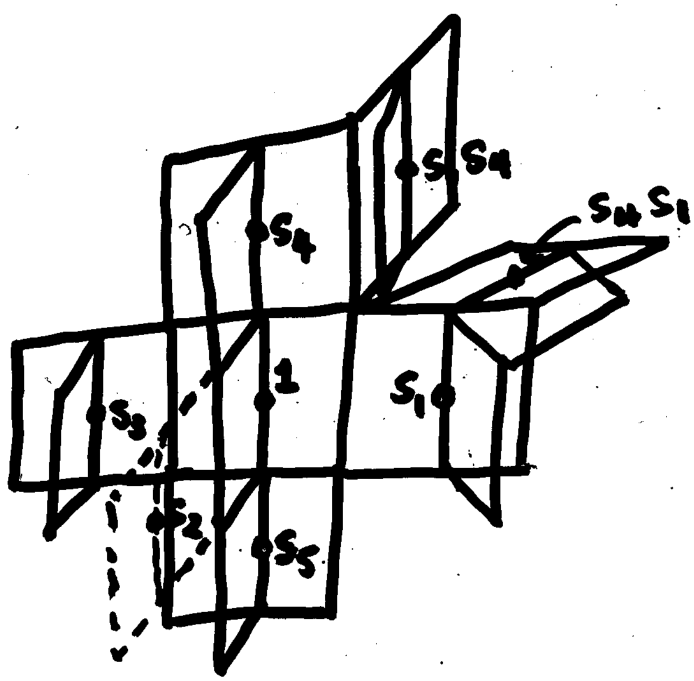
$C(L) =$ 

n-simplex in $L \longleftrightarrow (n+1)$-cube in $C(L)$

Davis-Moussong complex

$$X = C(L) \times W / \sim$$

Glue together W-many copies of C(L) along faces.



Links of vertices in X are L.

W acts on X prop. disconts. + cocompactly

$\Rightarrow$ W <u>uniform lattice</u> in Aut(X).

Nondiscreteness of Aut(X)

X Davis-Moussong complex for W

L = finite nerve of W = links of vertices in X.

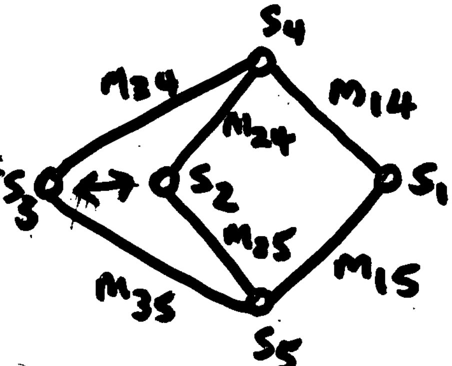
Theorem (Haglund-Paulin)

Aut(X) nondiscrete

$\Rightarrow \exists$ g in group of label-preserving automorphisms of L s.t. g fixes star of some $v \in \mathrm{Vert}(L)$ and $g \neq 1$.

e.g.

L =

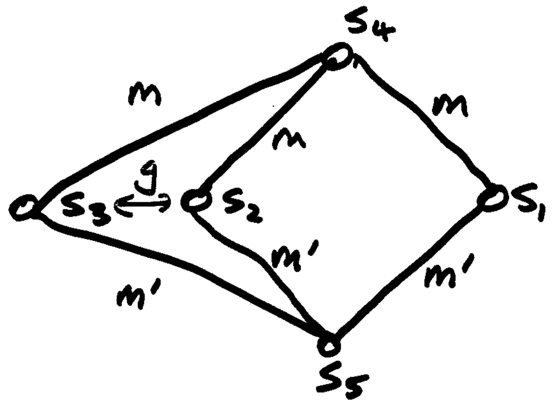


g fixes star(s_1)

g switches s_2 and s_3

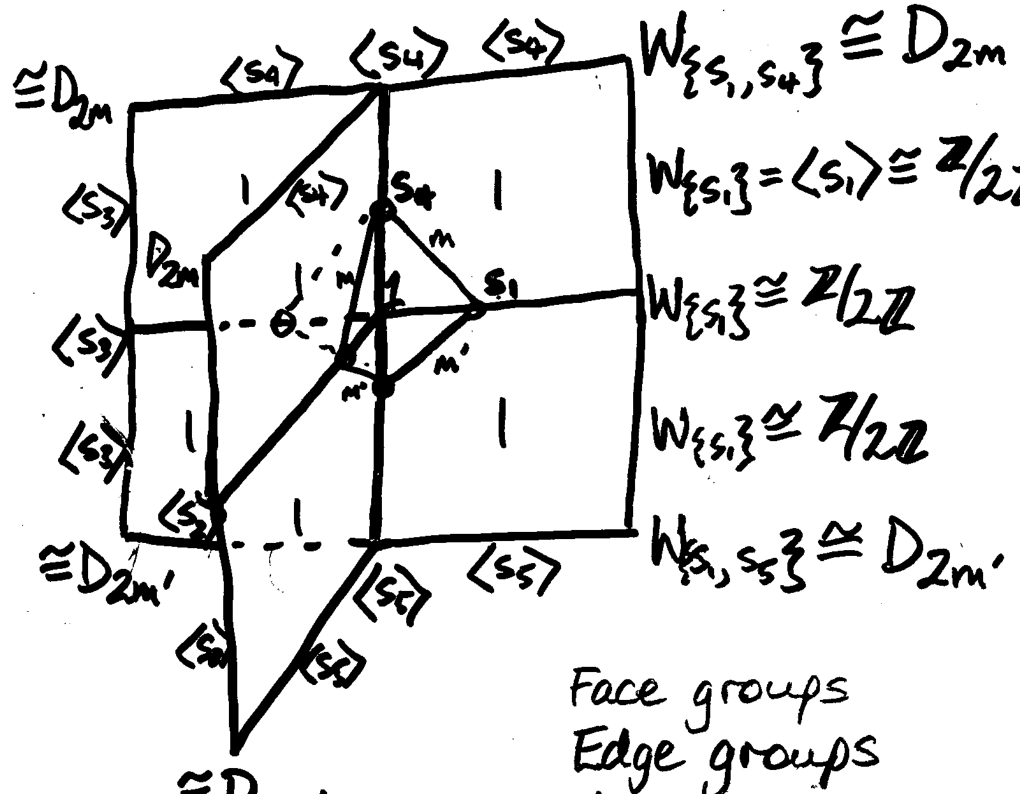
If $m_{24} = m_{34}$ and $m_{25} = m_{35}$ then Aut(X) is nondiscrete.

Construction of sequence of uniform lattices in Aut(X)

e.g. $L =$ 

m, m' even

Think of W as fund. gp of complex of groups over cubical cone $C(L)$:



Face groups
Edge groups
Vertex groups

The lattice Γ_1

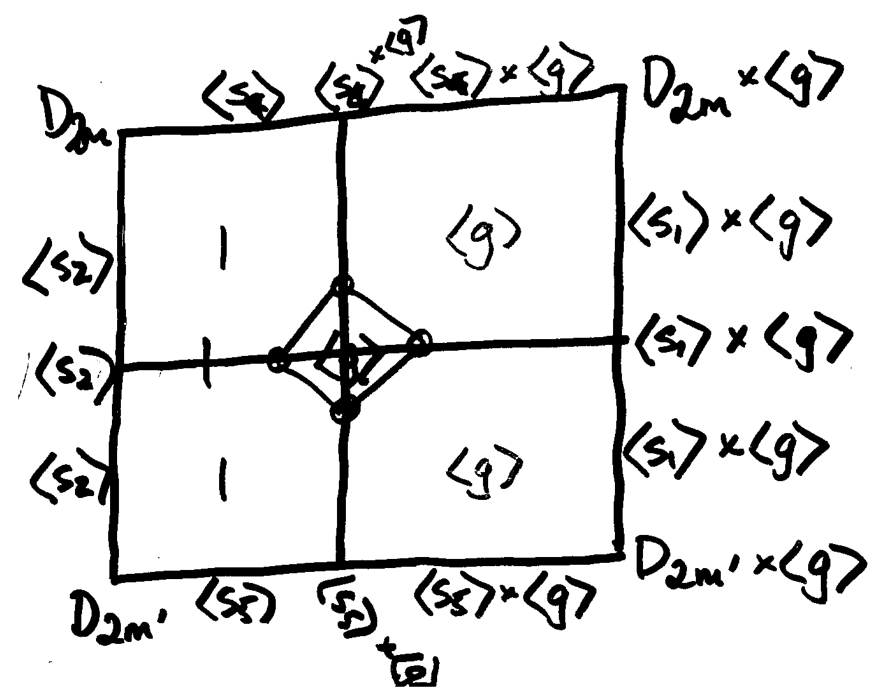
Aut(X) nondiscrete

$\Rightarrow \exists\ g \in \mathrm{Aut}(L)$ (label-preserving)
s.t. g fixes star of say $s_1 \in \mathrm{Vert}(L)$ and $g \neq 1$

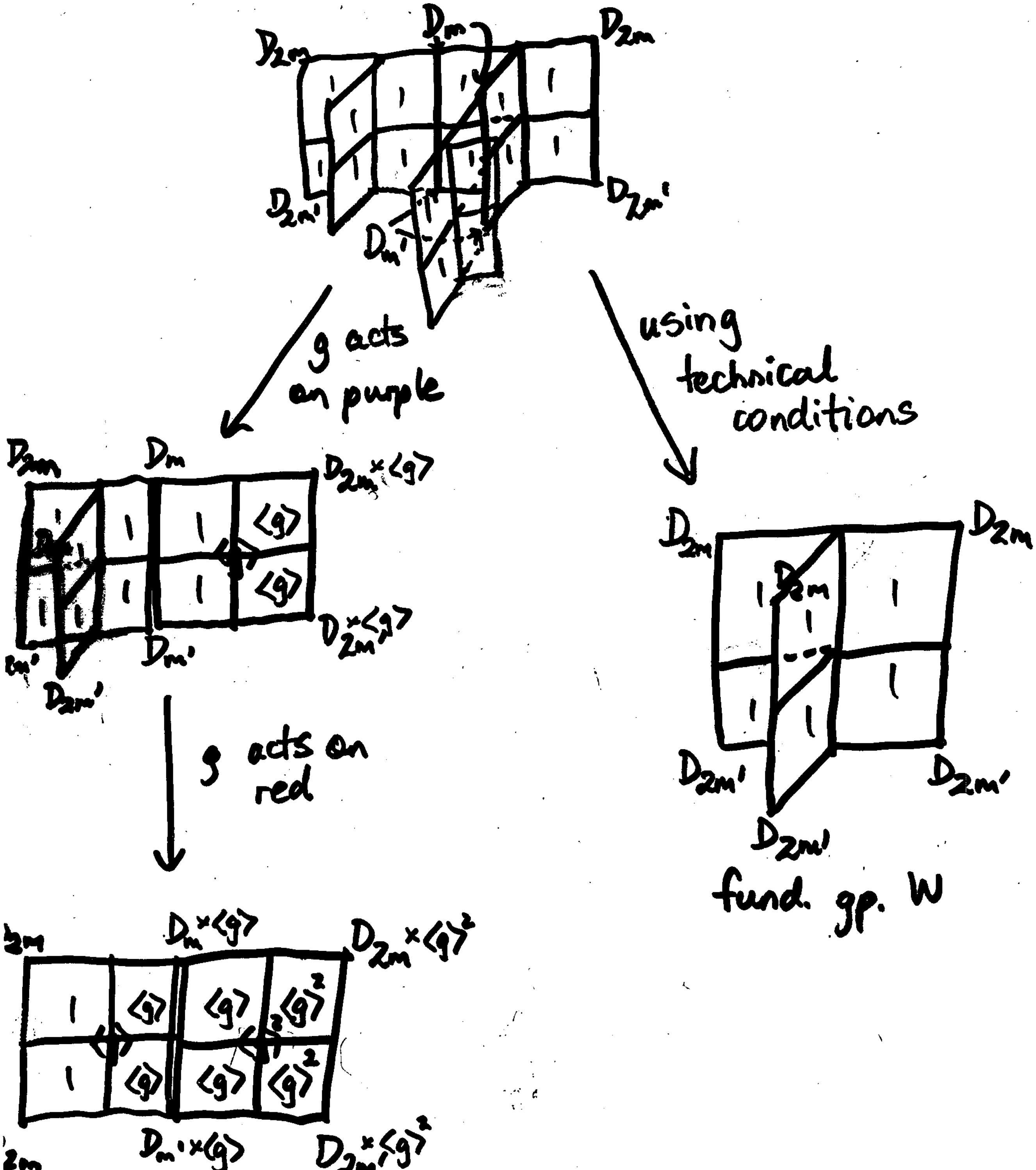
Wlog $|\langle g\rangle| = q$ prime, so for all s_i, $\mathrm{Stab}_{\langle g\rangle}(s_i) = \{1\}$ or $\langle g\rangle$.

Action of g extends to $C(L)$ and to complex of groups for W.

Get Γ_1 = fund. gp of resulting complex of groups:

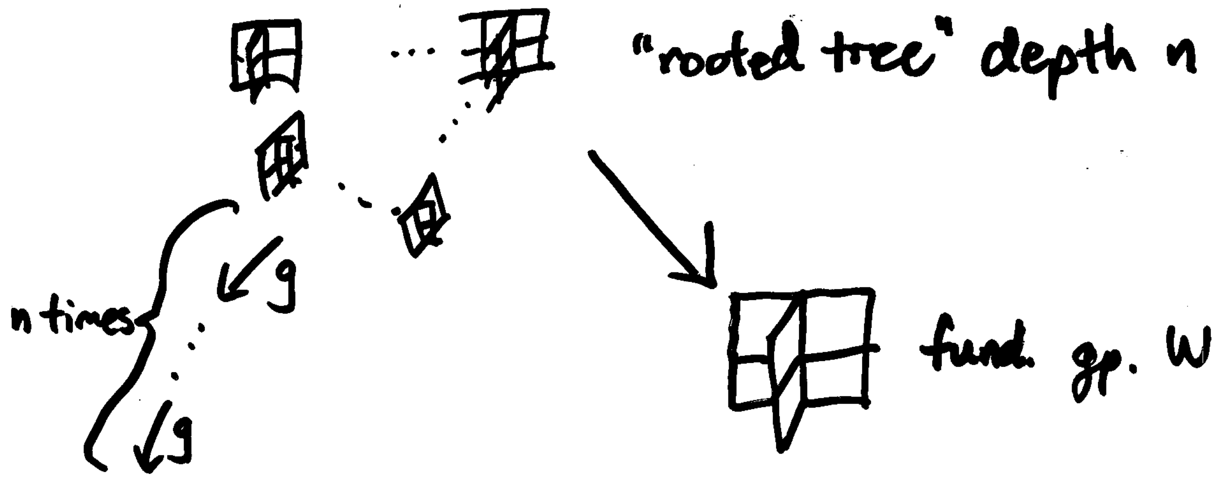


The lattice Γ_2



fund. gp. Γ_2
covolume $\frac{1}{4r} + \frac{1}{q^2}$

The lattices (Γ_n)



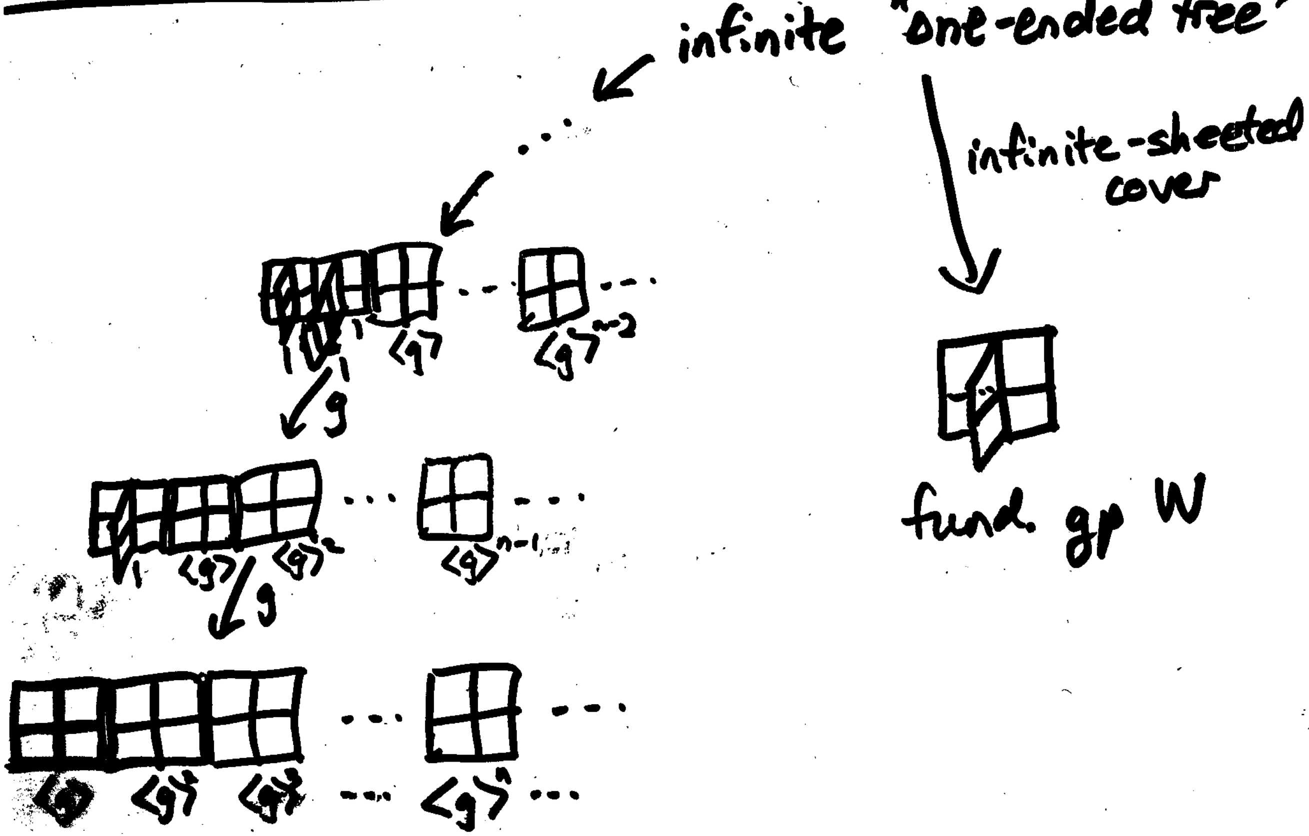
fund. gp $\qquad \Gamma_n$

covolume $\qquad \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^n}$

Coverings $\Rightarrow$ $\Gamma_n \leq Aut(X)$
unif. lattice

Remarks 1. all Γ_n commens. to W;
for some 2-dim X, all unif. lattices
are commens. to W (Haglund).
2. Moussong: conditions for X CAT(−1).
Get new hyperbolic gps, including dim > 2.

Nonuniform lattice Γ



fund. gp. Γ

covolume $\sum_{k=1}^{\infty} \frac{1}{q^k}$

Coverings $\Longrightarrow$ $\Gamma \leq Aut(X)$
nonunif. lattice

__Remark__ expect that variations on construction give many noncommens. nonunif. lattices, using invariants of Connell-Hruska.

Right-angled buildings

A Coxeter group W is right-angled if all $m_{ij} = 2$ or ∞.

e.g. $W = \langle s_1, \ldots, s_p \mid s_i^2 = (s_i s_{i+1})^2 = 1 \rangle$

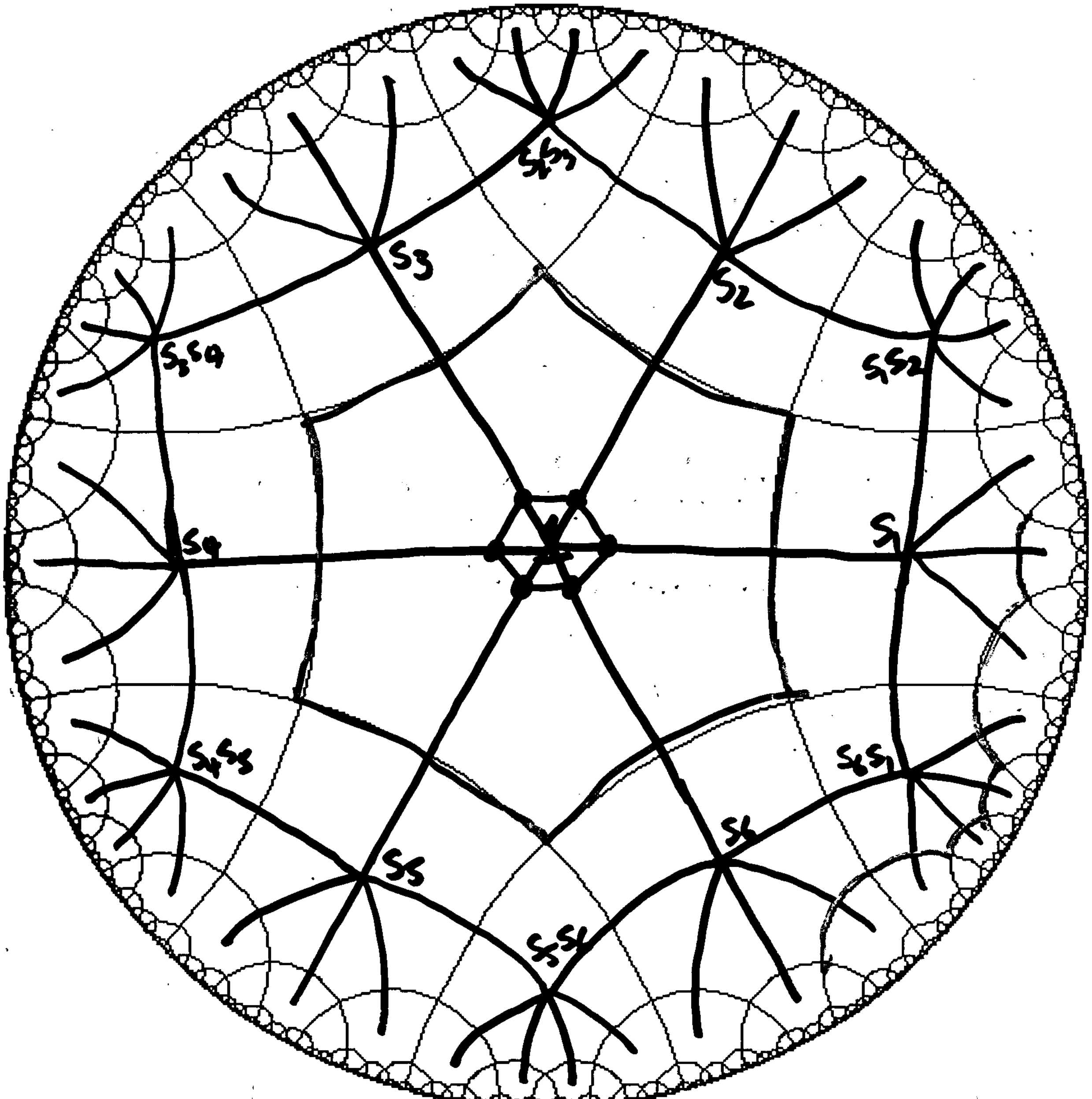
X a right-angled building has apts. Davis-Moussong complexes for W.

e.g. Bourdon's building $I_{p,q}$
apts modelled on W above
q = degree of branching at codim 1 faces

Aut(X) nondiscrete $\Longleftrightarrow$ X is "thick" building

e.g. $q \geq 3$

$I_{6,2}$



tile6-4.gif

Right-angled buildings

Some "higher-rank" behaviour:

<u>Theorem</u> (Bourdon–Pajot)
$I_{p,q}$ is quasi-isometrically rigid.

<u>BUT</u> "treelike" results too:

Haglund
T
Barnhill–T

Density of commensurators for right-angled buildings

Theorem (Haglund, Barnhill–T)
Let X be a right-angled building.
Let $\Gamma_0 \leq Aut(X)$ be the "standard unif. lattice".
Then $Comm_G(\Gamma_0)$ is dense in $G = Aut(X)$.

Pf.

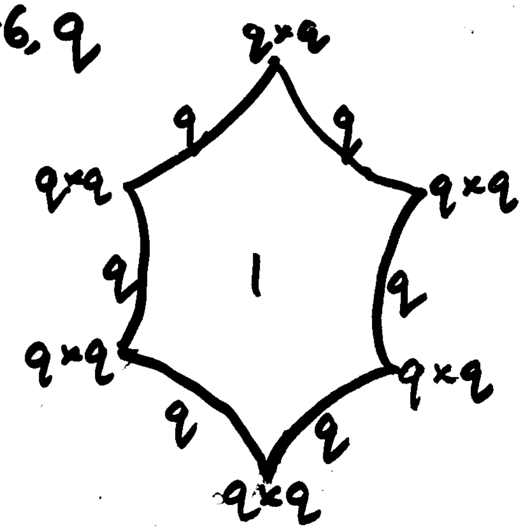
Theorem (Bass–Kulkarni, Liu)
Let T be a locally finite tree.
Let $\Gamma_0 \leq Aut(T)$ be "standard unif. lattice".
Then $Comm_G(\Gamma_0)$ is dense in $G = Aut(T)$.

(we give new simpler proof for T regular or biregular tree.)

The standard unif. lattice Γ_0

Fund gp of complex of finite cyclic gps and direct products of these.

e.g. $I_{6,q}$



Key property: $G_0 = \mathrm{Aut}_0(X)$ type-preserving automs.

$$\Gamma_0 \backslash X = G_0 \backslash X.$$

$\Rightarrow \forall\ x \in X$

$$G_0 = \mathrm{Comm}_{G_0}(\Gamma_0) \cdot \mathrm{Stab}_{G_0}(x)$$

$\Rightarrow$ enough to show

$\mathrm{Comm}_{G_0}(\Gamma_0)$ dense in $\mathrm{Stab}_{G_0}(x)$.

Want to show:

$\forall\ g \in Stab_{G_0}(x)$

$\forall\ k \geq 1$

there is a $\gamma \in Comm_{G_0}(\Gamma_0)$ s.t.

$$g|_{B_x(k)} = \gamma|_{B_x(k)}.$$

<u>Steps</u>

- g fixes some ball $B_x(r)$ $1 \leq r \leq k$

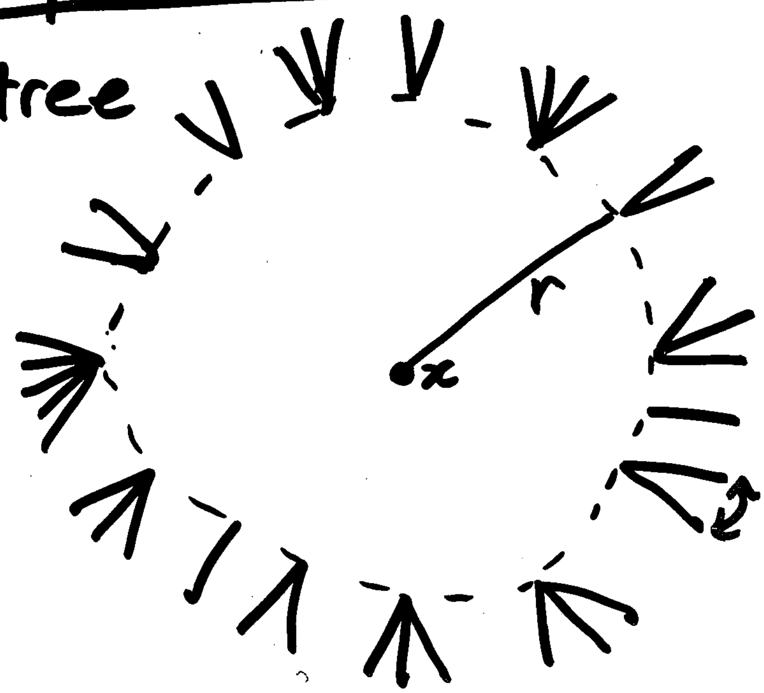
  Analyse action of g on $B_x(r+1)$: show can write g as product of <u>transpositions</u> (simpler automs.)

- approximate transpositions by elements of lattices commens. to Γ_0.

(construct <u>covering</u> of complexes of groups)

Transpositions

X = tree



transposition
- fixes $B_x(r)$
- switches 2 edges in $B_x(r+1)$

X = right-angled building

- switching 2 faces in $B_x(r+1)$ entails switching adjacent faces
- use product structure of Aut(links).