Note Taker Checklist Form -*MSRI*

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Talk Title and Workshop assigned to: Higher isoperimetric ~~inequality~~ inequalities for complexes and groups.

Lecturer (Full name): Panagiotis ~~[illegible]~~ Papazoglou

Date & Time of Event: Nov 6 2007 10:30 AM - 12:20 PM

Check List:

( ) Introduce yourself to the lecturer prior to lecture. Tell them that you will be the note taker, and that you will need to make copies of their own notes, if any.

( ) Obtain all presentation materials from lecturer (i.e. Power Point files, etc). This can be done either before the lecture is to begin or after the lecture; please make arrangements with the lecturer as to when you can do this.

( ) Take down all notes from media provided (blackboard, overhead, etc.)

( ) Gather all other lecture materials (i.e. Handouts, etc.)

( ) Scan all materials on PDF scanner in 2nd floor lab (assistance can be provided by Computing Staff) – Scan this sheet first, then materials. In the subject heading, enter the name of the speaker and date of their talk.

Please do **NOT** use **pencil** or colored pens other than black when taking notes as the scanner has a difficult time scanning pencil and other colors.

Please fill in the following after the lecture is done:

1. List 6-12 lecture keywords:

2. Please summarize the lecture in 5 or less sentences.

Once the materials on check list above are gathered, please scan ALL materials and send to the Computing Department. Return this form to Larry Patague, Head of Computing (rm 214)

X "reasonable" space, e.g. X riemannian mfd
or X simplicial complex

Def. Let $\gamma \subset X$ be a contractible loop.
We define the filling area of γ by:

$$FA_0(\gamma) = \inf_f \{ Area(f) \mid f : D^2 \to X,\ f(\partial D^2) = \gamma \}$$

Say X is simplicial complex, then we may assume f is simplicial

$$Area(f) = \# \{ \sigma \text{ 2-simplex of } D^2 \text{ s.t. } f(\sigma) \text{ is 2-simplex} \}$$

We define then the filling area function of X:
$$FA_0^X : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \cup \{\infty\}$$

$$FA_0^X(\ell) = \sup \{ FA_0(\gamma) : length(\gamma) \leq \ell \}$$

Often we assume X to be simply connected (e.g. pass to $\tilde{X}$)

Thm (Gromov '85) "Gap theorem"
Assume that $\exists \ell_0 : \forall \ell \geq \ell_0,\ FA_0^X(\ell) \leq \frac{1}{16\pi} \ell^2$.
Then $\exists c > 0$ s.t. $\forall \ell \geq \ell_0$ $FA_0^X(\ell) \leq c \cdot \ell$

In fact if $FA_0^X(\ell) \leq c\ell$ then $FA_0^X(\ell) \sim \ell$ unless X is a tree.

Other proofs: Olshanskii, Bowditch, P, Drutu

Wenger (2006): optimal constant $\frac{1-\varepsilon}{4\pi}$, $\varepsilon > 0$.

This thm has an interesting corollary in group theory, it gives a recognition algorithm for hyperbolic groups. One takes X = Cayley complex.

<u>Remark</u> Clearly there are no other "gaps" for $FA_0(\ell)$ e.g. add high "hills" to $\mathbb{E}^2$.

"Philosophical" remark

If there is a space with a certain geometric property then there is a group with the same property.

To put it differently it's too much to hope that something that holds for spaces does not hold for groups

Of course although we expect "random" properties of spaces to be satisfied also by groups it takes a lot of ingenuity to construct such examples

Birget - Olshanskii - Rips - Sapir
Bridson
Brady - Bridson

$\Rightarrow \exists$ no "gap" above ℓ^2 for Cayley complexes

One can define similarly higher dimensional filling functions. For example one can consider filling of 2-spheres by balls in X and define: $Vol_3^X : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \cup \{\infty\}$

Thm (P)
Let G be an FP_3 group.
Then Vol_3^G is bounded by a recursive function.
In fact Vol_4^G is also subrecursive.

Questions

1) Is Vol_n^G bounded by a recursive function for $n \geqslant 4$? (Yes if S^4 is algorithmically recognizable - but this is open)

2) Is Vol_3^G actually recursive?

3) If we consider filling of 2-cycles by 3-chains and define Vol_3^H is this fctn bounded by a recursive function? (Probably not)

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New guess: If $FV_2(n) \sim n^2$ and $\lim_{n\to\infty} \frac{FV_3(n)}{n^{3/2}} = 0$

then $\exists c > 0$ s.t. $FV_3(n) \leq cn$.

More generally one can consider "gaps" for the pair of functions $(FV_2(n), FV_3(n))$

e.g. is $(n^5, n\log n)$ possible?

<u>Def</u> A k-cycle S with $\mathrm{Vol}_k(S) = n$ is called a minimizer if $\mathrm{Fillvol}(S) = FV_{k+1}(n)$

Note: if $k=2$ a cycle S is a (singular) surface so we can define genus(S).

<u>Thm</u>

X simplicial complex s.t. $H_1(X) = H_2(X) = 0$

Assume that the following hold:

- $\exists K > 0$ s.t. $FV_2(n) \leq Kn^2$, $\forall n \in \mathbb{N}$

- $\lim_{n\to\infty} \frac{FV_3(n)}{n^{3/2}} = 0$

- There is some $g \in \mathbb{N}$ s.t. if S is minimizer 2-cycle then genus$(S) \leq g$

Then for every $\varepsilon > 0$ we have $\lim_{n\to\infty} \frac{FV_3(n)}{n^{1+\varepsilon}} = 0$

More generally:

if we assume instead that

- $FV_2(n) \le K n^r$, $K > 0$, $r \geq 2$
- same assumption on minimizers and
- $\lim\limits_{n \to \infty} \dfrac{FV_3(n)}{n^{\frac{2r-1}{2r-2}}} = 0$

we conclude that $\dfrac{FV_3(n)}{n^{1+\varepsilon}} \to 0 \quad (\forall \varepsilon > 0)$

For example (under the same assumption on minimizers) the pair $(n^4, n^{\frac{11}{10}})$ is not possible for $(FV_2(n), FV_3(n))$.

Gromov conjectures this thm holds w/o restriction on minimizers if X is CAT(0) with a discrete co-compact G-action

Geometric Parenthesis

M^n riem. mfld

Isoperimetric profile

$$I_M(t) = \inf_{\underline{\Omega}} \{ Vol_{n-1}(\partial \underline{\Omega}) : \underline{\Omega} \subset M^n \text{ open}, Vol_n(\underline{\Omega}) \leq t \}$$

roughly inverse of FV_n

e.g. for $n=2$ $\overset{S \approx \mathbb{R}^2}{\vee}$ Gromov's thm implies:

Assume $\exists t_0$ s.t. $I_S(t) \geqslant 4\sqrt{\pi}\sqrt{t}$ $\forall t \geqslant t_0$

Then $I_S(t) \geqslant \delta t$ $\forall t \geqslant t_0$. $(\delta > 0)$

Question

Let M^n be a non-positively curved mfld homeomorphic to $\mathbb{R}^n$.

Is it true that $I_M(t) \geqslant I_{\mathbb{E}^n}(t)$

($\mathbb{E}^n$ euclidean space)

True $n=3$ Kleiner, $n=4$ Croke, $n>4$ open

Thm $\Rightarrow$ (assuming bdd genus minimizers) there is a "gap" in the isoperimetric profile for $n=3$

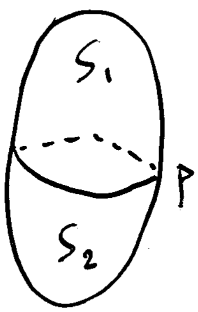
Thm (Hersch, Young-Yau, Korevaar, Lipton-Tarjan, Gilbert-Hutchinson-Tarjan) 80's

For any $g \in \mathbb{N}$ there is a constant $c(g) > 0$ s.t. if S is a surface of genus g and area $A(S)$ then the cheeger constant of S satisfies

$$h(S) \leq c(g) \frac{1}{\sqrt{A(S)}}$$

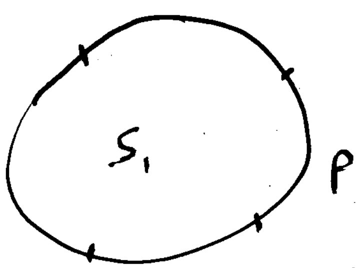
e.g. if $A(S) = 1$ $\quad h(S) \leq c(g)$

Proof (for S^2)



Let p be a shortest curve subdividing S in 2 regions S_1, S_2 s.t. $A(S_i) \geq \frac{1}{4} A(S)$ $(i = 1, 2)$

Assume $A(S_1) \geq A(S_2)$



There are no "shortcuts" in S_1 o.w. p wouldn't be smallest

Besicovitch lemma $\Rightarrow A(S_1) \geq \frac{\ell(p)^2}{16} \Rightarrow \ell(p) \leq 4\sqrt{A(S_1)}$

Def (Cheeger constant)

$$h(M) = \inf_{\substack{A \\ \text{open}}} \left\{ \frac{\mathrm{Vol}_{n-1}(\partial A)}{\mathrm{Vol}_n(A)} : \mathrm{Vol}_n(A) \leq \frac{1}{2} \mathrm{Vol}_n(M) \right.$$

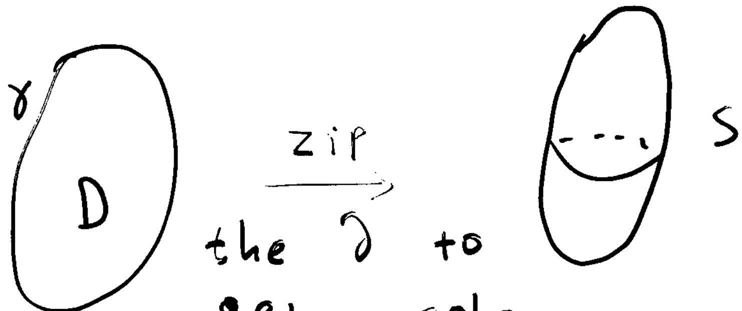
Proof of Gromov's "gap" theorem

Assume $FA_0^X(\ell) \leq \frac{1}{10^4} \ell^2, \ \forall \ell \geqslant \ell_0$

By contradiction: Say $\forall c > 0$

Take $c \gg \ell_0$

$\exists \gamma$ s.t. $FA_0(\gamma) > c \cdot \ell(\gamma)$. Assume γ minimal with this property.



the ∂ to get a sphere

Cheeger constant bound $\Rightarrow \exists p$ s.c.c. on S with $FA_0(p) \geqslant \frac{1}{16} \ell(p)^2$

Let β be a minimal length s.c.c. on S s.t. i) $\ell(\beta) \geqslant c$

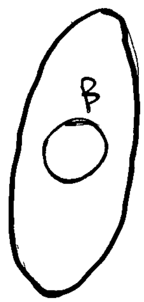
ii) $FA_0(\beta) \geqslant \frac{1}{100} \ell(\beta)^2$

Then β has no "short cuts" so

$: > \ell(\beta) > c.$

Lift now β back to D

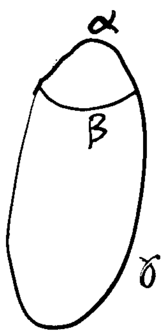
Either



in which case we are done

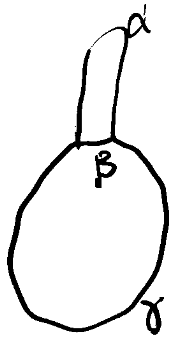
$FA_0(\beta) \sim \ell(\beta)^2$

or



again ok if $\ell(\alpha) \lesssim \ell(\beta)$

but this



is impossible by the minimality of γ.

We actually proved something stronger

Define FA_g to be filling by a surface of genus $\leq g$.

<u>Thm</u> If $FA_g(\ell) < c_g \ell^2 \quad \forall \ell \geq \ell_0$

then $FA_g(\ell) < c \cdot \ell \quad \forall \ell \geq \ell_0$.

If X is simply connected then this follows from Gromov's thm

Proof of thm

We argue by contradiction, we assume that for some $\varepsilon > 0$ $\frac{FV_3(n)}{n^{1+\varepsilon}} \to \infty$ and we show that $\limsup_{n \to \infty} \frac{FV_3(n)}{n^{3/2}} > 0$.

Lemma

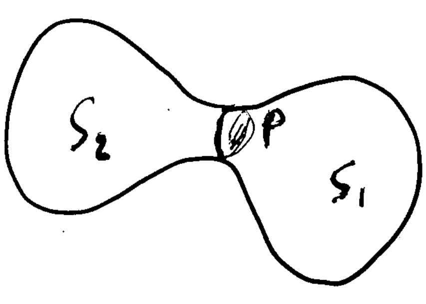
There is a $\delta > 0$ s.t. for any $M > 0$ if S is a minimal area 2-cycle with the property $\mathrm{FillV}(S) > M\, A(S)^{1+\varepsilon}$ then $\mathrm{diam}(S) > \delta \sqrt{A(S)}$

Pf

Assume again S is a 2-sphere.

There is a s.c.c. p separating S in S_1, S_2 s.t. $\ell(p) \sim \sqrt{A(S)}$ s' $A(S_i) \geq \frac{A(S)}{4}$.

If $\mathrm{diam}(S) < \delta\sqrt{A(S)}$ then $\mathrm{diam}(p) < \delta\sqrt{A(S)}$ and $\mathrm{FillV}_2(p) < k\delta A(S) \ll A(S)$ for δ "small"



We can cup off S_1, S_2 to $\tilde{S}_1, \tilde{S}_2$ with $A(\tilde{S}_i) \sim A(S_i)$ so

$$\mathrm{FillV}(S) \leq M\left(A(\tilde{S}_1)^{1+\varepsilon} + A(\tilde{S}_2)^{1+\varepsilon}\right) < M\, A(S)^{1+\varepsilon}$$

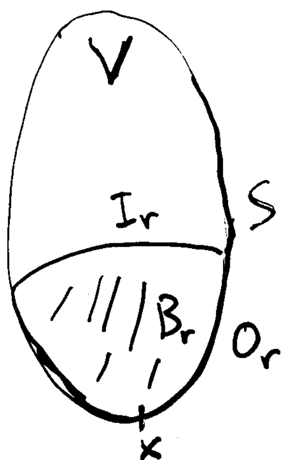
since $x^{1+\varepsilon}$ is convex. Contradiction

Let S be as in the Lemma. We construct a 2-cycle S_1 s.t.

$$\mathrm{FillV}(S_1) \sim A(S_1)^{3/2}.$$

To simplify assume S is an embedded sphere, $S = \partial V$

Fix $x \in S$ and consider $B_x(r)$, $r > 0$



Set $O_r = B_x(r) \cap S$

$I_r = \partial B_x(r) - O_r$

As in the lemma $I_r \gtrsim O_r$

Integrate $\displaystyle\int_{\delta_n/4}^{\delta_n/2} I_r$

Remark: most of the time $I_r < r^2$ since $\mathrm{FillV}(S) << n^{3/2}$

<u>Elementary inequality</u>

If $F(s) = \int_a^s f(t)\,dt$ and i) $a < f(t) < a^2$

ii) $F(2a) < \mu a^3$

then for some r, $F(r) > \frac{1}{20\mu} f(r)^{3/2}$